An 81/29 Upper Bound for Tuza's Conjecture via Helper-Arborescence Coupling and Opposite-Edge Elimination

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Draft

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Abstract

For a finite simple graph G, let $\nu(G)$ be the maximum number of pairwise edge-disjoint triangles and $\tau(G)$ the minimum number of edges meeting every triangle. Tuza conjectured in 1981 that $\tau(G) \le 2\nu(G)$, whereas taking every edge of a maximum packing gives $\tau(G) \le 3\nu(G)$. Building on Haxell's 1999 bound $\frac{66}{23}\nu(G)$, Sichen Wang (`arXiv:2609.13831`, September 2026) proved $\tau(G) \le \frac{165}{59}\nu(G) \approx 2.796610\,\nu(G)$ with an MIT-licensed Lean 4 formalization (`sichen-wang/tuza-lean`).

Extending Wang's framework and Lean 4 library under identical Mathlib `SimpleGraph` invariants `nu` and `tau`, we sharpen the universal constant in two stages. First, helper-arborescence coupling ($+\frac{3}{8}h \to +\frac{1}{4}\hat{h} \le +\frac{1}{4}h$) and unmerged colored-family covers give $250\,\tau(G) \le 699\,\nu(G)$ ($\tau(G) \le 2.796\,\nu(G)$), matching the infinite-depth Lemma 2.3 LP optimum while exposing a 32-vertex colored gadget and a whole-graph opposite-edge obstruction. Second, promoting that obstruction to all finite simple graphs eliminates every non-private opposite edge in Wang's book cover and yields $29\,\tau(G) \le 81\,\nu(G)$, i.e., $\tau(G) \le \frac{81}{29}\nu(G) \approx 2.793103\,\nu(G)$, saving $\frac{6}{1711}\nu(G)$ over Wang's bound. Theorems 1, 2(a), and 3 are verified unconditionally in Lean 4 (`lean/TuzaBound`, 28 files, 113 receipted of 544 public declarations, zero `sorry` or `native_decide`); the 32-vertex gadget and enumerations are verified in `recompute.py`. Unread sources are marked [to check].

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1 Introduction and main results

In any finite simple graph $G = (V, E)$, let $\nu(G)$ be the maximum size of a family of pairwise edge-disjoint triangles and $\tau(G)$ the minimum size of an edge subset meeting every triangle of G. Since every triangle of G shares at least one edge with a maximum packing P of size $\nu(G)$, taking all $3\nu(G)$ edges of P yields the trivial bound $\tau(G) \le 3\nu(G)$. In 1981, Zsolt Tuza [17] [to check primary 1981 and 1990 texts] conjectured that two edges per packed triangle always suffice—also recorded as Erdős Problem #167 [3, 4]:

$$\tau(G) \le 2\nu(G) \qquad \text{for every finite simple graph } G,$$

which is sharp on K_4 ($\nu = 1, \tau = 2$) and K_5 ($\nu = 2, \tau = 4$).

Krivelevich [11] [to check printed article] proved both fractional halves $\tau(G) \le 2\nu^*(G)$ and $\nu^*(G) \le 2\nu(G)$ in 1995, where $\nu^*(G) = \tau^*(G)$ is the fractional triangle-packing and covering optimum. For the integral ratio $\tau(G)/\nu(G)$ on arbitrary finite simple graphs, three advances improved upon 3 prior to the present work:

1. **Haxell (1999) [6] [to check printed article]:** $\tau(G) \le \frac{66}{23}\nu(G) \approx 2.869565\,\nu(G)$ (sketching $\frac{1+\sqrt{481}}{8}\nu(G) \approx 2.866464\,\nu(G)$) via a multi-stage packing decomposition (P, A, P') and random vertex bipartition.
2. **Yi (August 2026) [21]:** $\tau(G) \le \frac{63}{22}\nu(G) \approx 2.863636\,\nu(G)$ (and $\frac{162+4\sqrt{3}}{59}\nu(G) \approx 2.863190\,\nu(G)$) via 2-colorable triangle families, noting that Haxell's four-inequality framework alone hits a barrier at $\frac{54}{19}\nu(G) \approx 2.842105\,\nu(G)$.
3. **Sichen Wang (September 2026) [18, 20]:** In a major breakthrough below 2.8, Sichen Wang [18] observed that in the reduced graph $G' = G - E(A)$ (with remaining old edges O' colored blue and new edges red), the residual family $\mathcal{S} = \mathcal{F}(G' - B(P'))$ consists *entirely* of triangles with one red and two blue edges. Combining a private-spine exchange lemma (Lemma 2.1 of [18]), common-spine book covers of size at most 3 (Lemma 2.2), a two-page book random cut (4), a three-page book cover (5) on (P', Q, Q_1), and a merged colored-family estimate (6) on $\mathcal{S}$ (Lemma 2.3) with weights $(20, 8, 19, 12)$, Wang proved

$$59\,\tau(G) \le 165\,\nu(G) \qquad \Big(\tau(G) \le \frac{165}{59}\nu(G) \approx 2.796610\,\nu(G)\Big) \tag{1.1}$$

and released an MIT-licensed Lean 4 formalization (`sichen-wang/tuza-lean` [20]).

Contribution and credit to Wang. This work builds directly on Sichen Wang's combinatorial framework in [18] and ports and extends Wang's MIT-licensed Lean 4 library `sichen-wang/tuza-lean` [20] under character-for-character identical Mathlib `SimpleGraph` invariants `nu` and `tau`. Guided by an exact linear-programming audit of Wang's system (Proposition 3.5), we develop two progressive universal improvements. First, helper-arborescence coupling (Lemma 1.1) and unmerged colored-family bounds (Lemma 1.2)—neither of which improves (1.1) alone (Section 4.1)—lower the universal constant to $\frac{699}{250} = 2.796$ (Theorem 1), which we prove is the exact optimum of the infinite-depth Lemma 2.3 LP relaxation while isolating an explicit 32-vertex $\mathcal{S}$-gadget $H_{14,3,1}$ and a whole-graph opposite-edge obstruction (Theorem 2). Second, promoting that opposite-edge obstruction to every finite simple graph (Lemma 1.3) eliminates all non-private opposite

edges from Wang's two-page book cover and lowers the universal constant to $\frac{81}{29} \approx 2.793103$ using elementary non-recursive $\mathcal{S}$-covers alone (Theorem 3).

1.1 Statements of the main results

For any finite simple graph $G = (V, E)$, let (P, A, P', Q, Q_1) be a Haxell–Wang packing tuple (defined in Section 2) with cardinalities $n := |P| = \nu(G)$, $a := |A|$, $m := |P'|$, $b := |B^*|$ ($B^* = P' \cap \mathcal{F}_2$), $c := |Q|$, $d := |Q_1|$, and private-red-spine counts h in $\mathcal{S} = \mathcal{F}(G' - B(P'))$ and h_1 in $\mathcal{S}_Q = \mathcal{F}(G' - B(P') - B(Q))$. Partition the h private members of Q into $\mathcal{B}_{\text{out}}$ (two-page books whose opposite vertex pair is a red edge outside $N = R(B^*)$, of size $\hat{h} := |\mathcal{B}_{\text{out}}| \le h$) and $I_{\text{priv},0}$ (of size $h_0 := h - \hat{h} \ge 0$, so $\hat{h} + h_0 = h$).

Lemma 1.1 (Helper-arborescence coupling)**.** *For every finite simple graph G and every Haxell–Wang packing tuple (P, A, P', Q) on G, both the unrelaxed bound and the h-relaxed bound hold:*

$$4\,\tau(G) + 4c \le 6n + 10a + 6b + \hat{h}, \tag{4''}$$

$$4\,\tau(G) + 4c \le 6n + 10a + 6b + h. \tag{4'}$$

Lemma 1.1 sharpens Wang's $+\frac{3}{8}h$ random-cut term in [18, eq. (4)] to $+\frac{1}{4}\hat{h} \le +\frac{1}{4}h$ by organizing the ancestor basins of $\mathcal{B}_{\text{out}}$ under $j \to i$ ($\{x_j, y_j\} = e_i$) into pairwise disjoint sink-rooted in-arborescences, proving via a 2^r binary-toggle exchange that each spine e_v has at most $2 + 2\deg^-(v)$ uncovered-candidate apices, and summing the convex secant $\frac{1}{8}4^{-k} \ge \frac{1}{8} - \frac{3}{32}k$ over each arborescence.

Lemma 1.2 (Unmerged two-layer colored-cover decomposition)**.** *For every finite simple graph G and every Haxell–Wang packing tuple (P, A, P', Q, Q_1) on G, the two elementary non-recursive bounds and the recursive second-layer bound*

$$\tau(G) + b + 2d + h \le 3n + 3c + h_1, \tag{6a}$$

$$\tau(G) + b + h_1 \le 3n + 2c + 3d, \tag{6b}$$

$$3\,\tau(G) + 3b + 2h_1 \le 9n + 6c + 8d \tag{6c}$$

hold simultaneously on the same tuple (P, A, P', Q, Q_1).

Whereas [18] merged (6a) and (6c) in ratio $1 : 3$, keeping (6a)–(6c) separate lets us weight (6a) and (6c) in ratio $12 : 39 = 4 : 13$ to prove Theorem 1, and weight elementary (6a) and (6b) in ratio $2 : 4 = 1 : 2$ to prove Theorem 3.

Theorem 1 (Universal 2.796 upper bound)**.** *Every finite simple graph $G = (V, E)$ satisfies*

$$250\,\tau(G) \le 699\,\nu(G), \qquad \textit{equivalently,} \qquad \tau(G) \le \frac{699}{250}\,\nu(G) = 2.796\,\nu(G), \tag{1.2}$$

and every Haxell–Wang tuple (P, A, P', Q, Q_1) on G obeys the private-spine slack bound

$$500\,\tau(G) + 7h + 28h_1 \le 1398\,\nu(G). \tag{1.3}$$

Lean 4 verification of Theorem 1. Lemmas 1.1 and 1.2 and Theorem 1 ((1.2), (1.3), explicit edge-transversal existence `exists_small_triangle_transversal`, and Corollary 3.4 recovering $59\,\tau(G) \le 165\,\nu(G)$) are proved unconditionally in the Lean 4 kernel in `lean/TuzaBound` (`tuza_bound_nat`, `tuza_bound`; Section 5).

Theorem 2 (Infinite-depth LP fixed point and $\mathcal{S}$-layer barrier analysis)**.**

(a) ***(Infinite-depth LP fixed point — pure-kernel Lean 4 verified in `FixedPoint.lean`).*** *For every integer $K \ge 1$, consider the linear program maximizing τ/n over non-negative rationals n, τ, a, m, b and $(p_r, u_r)_{r=0}^{K}$ with $n > 0$ subject to $a + m \le n$, $b \le m$, $p_0 \le b$, $p_{r+1} + u_r \le p_r$ $(0 \le r < K)$, $u_K \le p_K$, disjoint blue-edge budget $a+3m-b+2\sum_{r=0}^{K} p_r \le 3n$, (3), (4′), (5), and all four Lemma 2.3 $\mathcal{S}$-layer bounds across $0 \le r \le K$ (namely $3p_r - u_r$ and $\frac{8}{3}p_r - \frac{2}{3}u_r$ for $0 \le r \le K$, $3p_r - 2p_{r+1} + u_{r+1} - u_r$ for $r+1 \le K$, and $3p_r - p_{r+1} - p_{r+2} - u_r$ for $r+2 \le K$, each shifted by $3a+3m-b+\sum_{j<r} 2p_j$). Its exact global optimum is*

$$\mathrm{OPT}_K = \frac{699}{250} \qquad \text{for every } K \ge 1,$$

and the infinite-depth system across all $r \in \mathbb{N}$ simultaneously has the same exact optimum $\frac{699}{250}$, attained at $(n, \tau, a, m, b) = (250, 699, 51, 199, 159)$, $(p_0, p_1, p_2) = (42, 9, 3)$, $(u_0, u_1, u_2) = (0, 0, 3)$, and $p_r = u_r = 0$ for $r \ge 3$.

(b) ***(Explicit $\mathcal{S}$-layer Lemma 2.3 gadget and whole-graph obstructions — proved in Section 3.3 and verified in `recompute.py`; not formalized in `lean/TuzaBound`).*** *There exists an explicit simple red-blue graph $H_{14,3,1}$ on 32 vertices and 62 edges (48 blue, 14 red) with 32 triangles (all having one red and two blue edges) and global invariants $\nu(H_{14,3,1}) = \tau^*(H_{14,3,1}) = \tau(H_{14,3,1}) = 14$ that realizes $(p_0, p_1, p_2) = (14, 3, 1)$ and $(u_0, u_1, u_2) = (0, 0, 1)$ with exact size 36 (zero LP slack) across all three recursive Lemma 2.3 $\mathcal{S}$-covers. Conversely, embedding three disjoint copies of $H_{14,3,1}$ into a finite whole graph at $n = 250$ faces two strict obstructions:*

- *(i) on every finite graph G, $\mathbb{P}[e \in D_{\mathrm{red}}] = \frac{1}{2}(1 - 2^{-|S_e|}) < \frac{1}{2}$ strictly for every $e \in N$, and if the $b - c = 117$ one-page spines of B_1^* have $S_e = \emptyset$, the expected random-cut size is at most $\frac{1281}{2} = 640.5 < 699$;*
- *(ii) under lexicographically maximal P', none of the 10 tail spines per copy of $H_{14,3,1}$ (30 spines across three copies) can carry an opposite edge outside $C^{(5)}_{\mathrm{base}}$, capping $|C^{(5)}| \le 699 - 30 = 669 < 699$.*

Scope fence on Theorem 2. Theorem 2(a) gives the exact optimum $\frac{699}{250}$ of the pre-(5″) *linear-programming relaxation*, not $\tau(G)/\nu(G)$ of a finite graph (Tuza's conjecture predicts $\tau(G)/\nu(G) \le 2$). Theorem 2(a) is kernel-verified in `lean/TuzaBound/FixedPoint.lean` (`infinite_depth_lp_exact_699_250`), whereas Theorem 2(b) is proved in Section 3.3 and verified in exact rational arithmetic by `papers/tuza-triangle-cover/recompute.py`, outside `lean/TuzaBound`.

Promoting Theorem 2(b)(ii)'s tail-spine exchange obstruction to arbitrary finite simple graphs yields an all-graph elimination of non-private opposite edges in Wang's book cover and lowers the universal constant to $\frac{81}{29}$:

Lemma 1.3 (Complete non-private opposite-edge elimination)**.** *For every finite simple graph G and every Haxell–Wang packing tuple (P, A, P', Q) on G, with $\hat{h} + h_0 = h$ as above,*

$$\tau(G) + 2c \le 3n + \hat{h} + 2h_0. \tag{5''}$$

Theorem 3 (Universal 81/29 upper bound)**.** *Every finite simple graph $G = (V, E)$ satisfies*

$$29\,\tau(G) \le 81\,\nu(G), \qquad \textit{equivalently,} \qquad \tau(G) \le \frac{81}{29}\,\nu(G) \approx 2.793103\,\nu(G), \tag{1.4}$$

saving $\frac{6}{1711}\nu(G)$ over Wang's $\frac{165}{59}\nu(G)$ and $\frac{21}{7250}\nu(G)$ over $\frac{699}{250}\nu(G)$, and every Haxell–Wang tuple (P, A, P', Q, Q_1) on G obeys the slack bound

$$29\,\tau(G) + 2h_1 \le 81\,\nu(G). \tag{1.5}$$

Lean 4 verification of Lemma 1.3 and Theorem 3. Both Lemma 1.3 (`Construction.opposite_elimination_cover` in `OppositeElimination.lean`) and Theorem 3 (`tuza_bound_81_29_nat`, `tuza_bound_81_29`, `exists_small_triangle_transversal_81_29`, and `Construction.bound_81_29_with_slack` in `Main.lean`) are proved unconditionally in the Lean 4 kernel using only elementary non-recursive $\mathcal{S}$-covers (6a) and (6b) (Section 5).

The 10-line Python snippet below checks in exact rational arithmetic both cleared integer conic certificates—$(170, 17, 160, 24, 26)$ proving (1.3) and $(10, 1, 8, 1, 2, 4)$ proving (1.5)—together with the saving $\frac{6}{1711}$ over $\frac{165}{59}$ (executed verbatim by `recompute.py`):

Concept in Action (Exact integer conic certificates for $500\,\tau + 7h + 28h_1 \le 1398\,\nu$ and $29\,\tau + 2h_1 \le 81\,\nu$).

```
from fractions import Fraction as F
M5 = [(3,-1,0,0,0,0,0), (6,10,6,-4,0,1,0), (3,0,0,-1,-1,0,0),
      (3,0,-1,3,-2,-1,1), (9,0,-3,6,8,0,-2)]  # (3),(4'),(5),(6a),(6c)
assert [sum(w*r[j] for w,r in zip((170,17,160,24,26), M5))
        for j in range(7)] == [1398, 0, 0, 0, 0, -7, -28]  # 500*tau<=1398*nu
M6 = [(3,-1,0,0,0,0,0,0), (6,10,6,-4,0,1,0,0), (3,0,0,-1,-1,0,0,0),
      (3,0,0,-2,0,1,2,0), (3,0,-1,3,-2,-1,-1,1), (3,0,-1,2,3,0,0,-1)]
assert [sum(w*r[j] for w,r in zip((10,1,8,1,2,4), M6))
        for j in range(8)] == [81, 0, 0, 0, 0, 0, 0, -2]   # 29*tau<=81*nu
assert F(165, 59) > F(699, 250) > F(81, 29) == F(165, 59) - F(6, 1711)
```

Figure 1 summarizes the mathematical pipeline and formal trust boundary. Section 2 states exact definitions matching `lean/TuzaBound`; Section 3 proves Lemmas 1.1–1.3 and Theorems 1–3; Section 4 presents exact LP certificates, finite arborescence and set-partition checks, and small-ν diagnostics; Section 5 documents the Lean 4 formalization; Section 6 covers prior work and open problems; and Section 7 records reproducibility and AI disclosures.

2 Definitions, Haxell–Wang packing decomposition, and setup

All graph-theoretic definitions below match verbatim `Basic/Geometry.lean`, `Basic/Packing.lean`, `Basic/Graph.lean`, `Construction/Setup.lean`, and `Construction/`

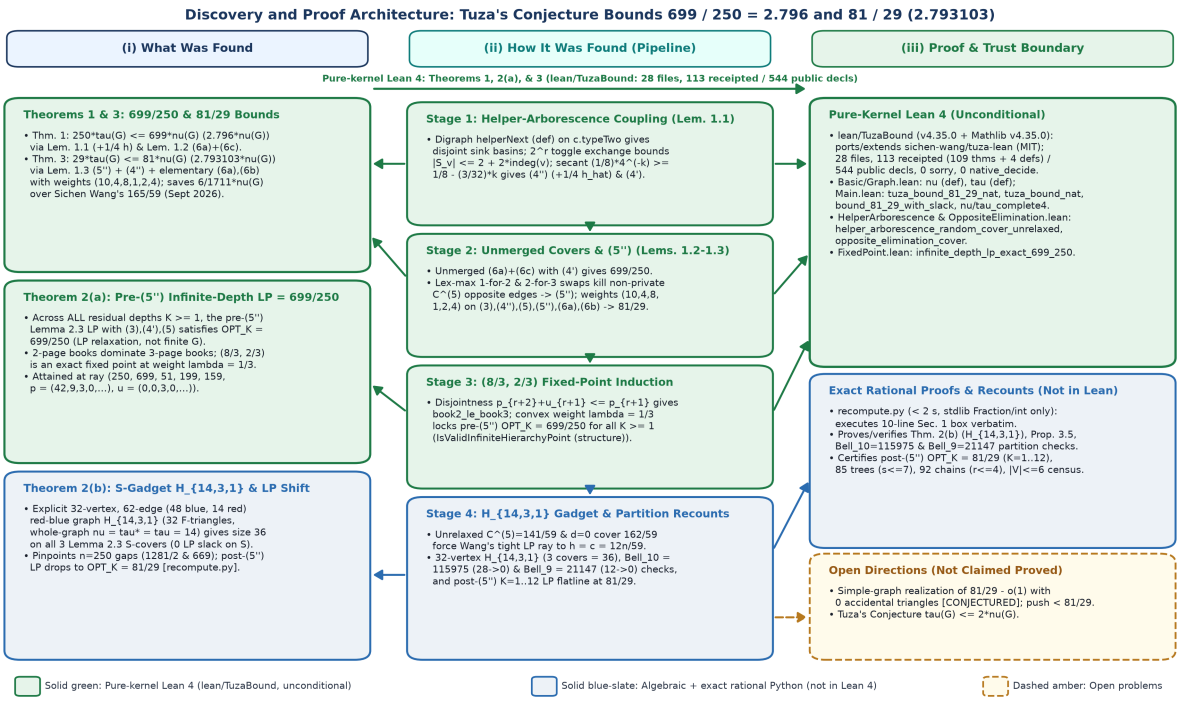


Figure 1: Discovery and proof architecture for the $\frac{699}{250} = 2.796$ and $\frac{81}{29} \approx 2.793103$ bounds on Tuza's conjecture. Column (i) states Theorems 1 and 3, Theorem 2(a) (pre-(5″) infinite-depth LP optimum $\mathrm{OPT}_K = \frac{699}{250}$), and Theorem 2(b) (32-vertex $\mathcal{S}$-gadget $H_{14,3,1}$ and LP shift to $\frac{81}{29}$). Column (ii) traces the four-stage pipeline, and Column (iii) separates pure-kernel Lean 4 proofs in `lean/TuzaBound` (solid green) from exact-rational Python verifications in `recompute.py` (solid blue-slate) and open problems (dashed amber).

`BookCover.lean` inside `lean/TuzaBound/`, adapted from Sichen Wang's `sichen-wang/tuza-lean` [20]. Throughout, $\mathbb{N} = \{0, 1, 2, \dots\}$, and $G = (V, E)$ is a finite simple graph with edge set $E = \mathrm{graphEdges}(G) \subseteq \binom{V}{2}$ represented as `Finset (Sym2 V)` via loopless unordered pairs $s(u, v)$ $(u \neq v)$.

2.1 Triangles, packings, transversals, and non-vacuity

Definition 2.1 (Triangles, packings, and transversals)**.** Let V be a finite set and $A \subseteq \binom{V}{2}$ a finite loopless edge set.

1. For distinct $u, v, w \in V$, $\mathrm{triangleEdges}(u, v, w) := \{uv, uw, vw\}$ satisfies `IsTriangle` ($|T| = 3$). Define
$$\mathrm{trianglesOn}(A) := \big\{T \subseteq A : \mathrm{IsTriangle}(T)\big\},$$
$$\mathcal{T}(G) = \mathrm{graphTriangles}(G) := \mathrm{trianglesOn}(\mathrm{graphEdges}(G)).$$
By `triangleEdges_mem_graphTriangles_iff`, $\mathrm{triangleEdges}(u, v, w) \in \mathcal{T}(G)$ holds if and only if $\{u, v, w\}$ is a 3-clique of G (`G.IsNClique 3 {u, v, w}`).
2. For a finite family $\mathcal{F}$ of edge sets, put $\mathrm{support}(\mathcal{F}) := \bigcup_{T \in \mathcal{F}} T$ (written $E(\mathcal{F})$) and $\mathrm{residual}(\mathcal{F}, C) := \{T \in \mathcal{F} : T \cap C = \emptyset\}$, which obeys $\mathrm{trianglesOn}(A \setminus C) =$

residual(trianglesOn$(A), C$) (`trianglesOn_sdiff`).

3. A subfamily $\mathcal{P} \subseteq \mathcal{F}$ is a *packing* (`IsPacking` $\mathcal{F}$ $\mathcal{P}$) if its members are pairwise edge-disjoint ($T_1 \cap T_2 = \emptyset$ for $T_1 \neq T_2$), and a *maximum packing* (`IsMaximumPacking` $\mathcal{F}$ $\mathcal{P}$) if it maximizes $|\mathcal{P}|$ among packings in $\mathcal{F}$, with maximum size packingNumber($\mathcal{F}$). Given $\mathcal{F}_0 \subseteq \mathcal{F}$, $\mathcal{P}$ is *lexicographically maximum* (`IsLexMaximumPacking` $\mathcal{F}$ $\mathcal{F}_0$ $\mathcal{P}$) if every packing $\mathcal{Q}$ in $\mathcal{F}$ satisfies either $|\mathcal{Q} \cap \mathcal{F}_0| < |\mathcal{P} \cap \mathcal{F}_0|$, or equality together with $|\mathcal{Q}| \leq |\mathcal{P}|$.
4. An edge set C is a *cover* (`IsCover` $\mathcal{F}$ C) if $T \cap C \neq \emptyset$ for all $T \in \mathcal{F}$, and transversalNumber($\mathcal{F}$) is the minimum of $|C|$ over covers of $\mathcal{F}$. Specializing to $\mathcal{T}(G)$ gives

$$\nu(G) = \mathrm{nu}(G) := \mathrm{packingNumber}(\mathcal{T}(G)),$$
$$\tau(G) = \mathrm{tau}(G) := \mathrm{transversalNumber}(\mathcal{T}(G)).$$

Because $\emptyset$ is always a packing and support($\mathcal{T}(G)$) $\subseteq E(G)$ is always a finite cover, neither $\nu(G)$ nor $\tau(G)$ can degenerate on empty sets; `exists_graph_minimumCover` and `graph_cover_iff_adj` certify that $\tau(G)$ is achieved by $C \subseteq E(G)$ meeting every 3-clique. On $K_4 = (\top : \mathrm{SimpleGraph}(\mathrm{Fin}\, 4))$, any two triangles share an edge (`complete4_triangles_not_disjoint`) and $\{01, 23\}$ is a minimum cover, certifying $\nu(K_4) = 1$ (`nu_complete4`) and $\tau(K_4) = 2$ (`tau_complete4`). The fractional LP optimum over $\mathbb{Q}$ is

$$\begin{aligned}\nu^*(G) &:= \max\Big\{ \sum_{T \in \mathcal{T}(G)} y_T : y \geq 0,\ \sum_{T \ni e} y_T \leq 1\ \forall e \in E \Big\} \\ &= \min\Big\{ \sum_{e \in E} x_e : x \geq 0,\ \sum_{e \in T} x_e \geq 1\ \forall T \in \mathcal{T}(G) \Big\} =: \tau^*(G), \end{aligned} \tag{2.1}$$

satisfying $\nu(G) \leq \nu^*(G) = \tau^*(G) \leq \tau(G) \leq 3\nu(G)$.

2.2 Books, colored triangle families, and Wang's foundational lemmas

Following Wang [18, 20] (`Basic/Geometry.lean`, `Books/Cover.lean`), a *book* is a collection of 1, 2, or 3 triangles (*pages*) sharing a common *spine*:

- `Book.single` $u\ v\ x$: one page uvx on three distinct vertices, with spine uv.
- `Book.double` $u\ v\ x\ y$: two pages uvx, uvy on four distinct vertices, with spine uv and *opposite vertex pair* $\{x, y\}$ (`Book.opposite` = some $s(x, y)$). Any triangle distinct from both pages that meets both pages while avoiding uv equals uxy or vxy and hence contains xy (`two_page_triangle`, `two_page_opposite_mem`).
- `Book.triple` $u\ v\ x\ y\ z$: three pages uvx, uvy, uvz on five distinct vertices; any triangle meeting all three pages must contain the spine uv (`three_page_spine`).

For a graph H partitioned into blue edges and red edges R, write $\mathcal{F}(H)$ (`oneRedTriangles` on $(E(H), R)$) for the family of triangles of H having *one red edge and two blue edges* (`ColoredFamily`). For $\mathcal{P} \subseteq \mathcal{F}(H)$, put $R(\mathcal{P}) := E(\mathcal{P}) \cap R$ and $B(\mathcal{P}) := E(\mathcal{P}) \setminus R$ (when $\mathcal{P}$ is a packing, $|R(\mathcal{P})| = |\mathcal{P}|$ and $|B(\mathcal{P})| = 2|\mathcal{P}|$). A member $T \in \mathcal{P}$ has a *private red edge* in $\mathcal{F}(H)$ (`privateRedPages` on $(\mathcal{F}(H), R, \mathcal{P})$) if no $W \in \mathcal{F}(H) \setminus \{T\}$ shares its red edge. Wang's Section 2 lemmas [18] read as follows:

Lemma 2.1 (Private-spine exchange, Wang [18, Lemma 2.1]; `unique_residual_triangle`)**.** *Let $P_{\mathcal{F}}$ be a maximum packing in $\mathcal{F}(H)$, and let $T = uvx \in P_{\mathcal{F}}$ have red spine $uv \in R(P_{\mathcal{F}})$ and second page $Q^* = uvy \in \mathcal{F}(H - B(P_{\mathcal{F}}))$ $(y \neq x)$. If xy is a red edge of H outside $R(P_{\mathcal{F}})$, then Q^* is the unique member of $\mathcal{F}(H - B(P_{\mathcal{F}}))$ containing uv.*

Lemma 2.2 (Common-spine book cover, Wang [18, Lemma 2.2]; `exists_cover_of_small_common_spine_books`)**.** *Let $\{\mathcal{B}_i\}_{i\in I}$ be pairwise edge-disjoint books in a triangle family $\mathcal{F}$ on E_0, each with $1 \le |\mathcal{B}_i| \le 3$ pages on spine e_i, such that every page-choice transversal has support meeting every member of $\mathcal{F}$. Writing $I_k := \{i \in I : |\mathcal{B}_i| = k\}$, taking $C_{\mathrm{base}} := \bigcup_{i\in I_1} E(\mathcal{B}_i) \cup \{e_i : i \in I\}$ plus each opposite edge $x_iy_i \in E_0 \setminus C_{\mathrm{base}}$ $(i \in I_2)$ covers $\mathcal{F}$ with size at most $3|I_1| + 2|I_2| + |I_3| = 4|I| - \sum_{i\in I}|\mathcal{B}_i|$.*

Lemma 2.3 (Colored residual covers, Wang [18, Lemma 2.3]; `first_colored_cover`, `colored_bound`, `recursive_colored_cover`)**.** *Let $P_{\mathcal{F}}$ be a maximum packing in $\mathcal{F}(H)$ of size p with u private-red members in $\mathcal{F}(H)$, and let $Q_{\mathcal{F}}$ be a maximum packing in $\mathcal{F}_1 := \mathcal{F}(H - B(P_{\mathcal{F}}))$ of size q with $r \le q$ private-red members in $\mathcal{F}_1$. Then:*

(i) Lemma 2.2 on $(P_{\mathcal{F}}, Q_{\mathcal{F}})$ (limiting opposite red edges to the r private members of $Q_{\mathcal{F}}$ via Lemma 2.1, and omitting the u private red spines of $P_{\mathcal{F}}$ since their two blue edges in C_{base} already cover them) yields $\tau(\mathcal{F}(H)) \le 3p - 2q + r - u$.

(ii) Combining $\frac{1}{3}$ of (i) with $\frac{2}{3}$ of $\tau(\mathcal{F}(H)) \le 2p + 3q - r$ ($B(P_{\mathcal{F}})$ plus $E(Q_{\mathcal{F}})$ minus r private spines) using $r \le q$ gives $3\,\tau(\mathcal{F}(H)) + 2u \le 8p$.

(iii) Deleting $B(P_{\mathcal{F}})$ ($2p$ edges) and applying (ii) to $(\mathcal{F}_1, Q_{\mathcal{F}})$ gives $3\,\tau(\mathcal{F}(H)) + 2r \le 6p + 8q$.

2.3 The Haxell–Wang packing tuple (P, A, P', Q, Q_1)

On any finite simple graph $G = (V, E)$, we form `c : Construction (graphEdges G)` and `r : RefinedConstruction c` (`exists_construction`, `exists_refinedConstruction`) in five stages:

1. **Packing P and types:** Fix a maximum packing P (`c.initial`) in $\mathcal{T}(G)$ of size $n := |P| = \nu(G)$; call $O := E(P)$ (`c.old`, $|O| = 3n$) *old* and $E \setminus O$ *new*. Every $T \in \mathcal{T}(G)$ has type $j := |T \cap O| \in \{1, 2, 3\}$.
2. **Type-1 packing A and (3):** Fix a maximum type-1 packing A (`c.first`) of size $a := |A| \le n$. Lemma 2.2 on the n books formed by P and A (`Construction.first_cover`) gives
$$\tau(G) + a \le 3n. \tag{3}$$
3. **Reduced graph G' and lexicographic packing P':** Set $E(G') := E \setminus E(A)$ (`c.remainingEdges`) and $O' := O \setminus E(A)$ (`c.oldRemaining`, $|O'| = 3n - a$). Every triangle of G' has type 2 or 3. Color O' blue and $E(G') \setminus O'$ red, and put $\mathcal{F}_2 := \mathcal{F}(G')$. Fix a packing P' (`c.rest`) in $\mathcal{T}(G')$ lexicographically maximum with respect to $\mathcal{F}_2$, maximizing first $b := |B^*|$ ($B^* := P' \cap \mathcal{F}_2 = \nu(\mathcal{F}_2)$-packing, `c.typeTwo`) and second $m := |P'|$ ($a + m \le n$). Set $N := R(B^*) = E(P') \setminus O$ (`c.newEdges`, $|N| = b$) and $B(P') := E(P') \cap O'$ ($|B(P')| = 3m - b$).
4. **Residual family $\mathcal{S}$ and packing Q:** Any triangle of G' avoiding $B(P')$ cannot have type 3 (else adding it to P' yields $(b, m + 1)$) and must have its red edge in N (by

maximality of B^*), so deleting $B(P')$ leaves $\mathcal{S} := \mathcal{F}(G' - B(P'))$ (`c.specialFamily`, `specialFamily_eq_residual`). Fix a maximum packing Q (`r.second`) in $\mathcal{S}$ of size $c := |Q| \le b$, with $h := |\mathrm{privateRedPages}(\mathcal{S}, N, Q)|$.

5. **Packing Q_1 and (5):** Fix a maximum packing Q_1 (`r.third`) in $\mathcal{S}_Q := \mathcal{F}(G' - B(P') - B(Q))$ of size $d := |Q_1| \le c - h$, with $h_1 := |\mathrm{privateRedPages}(\mathcal{S}_Q, N, Q_1)| \le d$. Applying Lemma 2.2 to the m books (P', Q, Q_1) in G' ($m + c + d$ pages) and adding $E(A)$ (`Construction.full_book_cover`) gives

$$\tau(G) \le 3a + 3m - c - d \le 3n - c - d, \qquad \text{i.e.,} \qquad \tau(G) + c + d \le 3n. \tag{5}$$

3 Proof of the main results

3.1 Helper-arborescence coupling (Proof of Lemma 1.1)

Fix a Haxell–Wang tuple (P, A, P', Q) on G and follow `Probability/Arborescence.lean` and `Construction/HelperArborescence.lean`. Pairing each $Q^* \in Q$ with the unique parent $T \in B^*$ sharing its red spine in N splits B^* into c two-page books $\mathcal{B}_i = \{T_i, Q_i^*\}$ ($i \in I_2$, where $T_i = u_i v_i x_i$ and $Q_i^* = u_i v_i y_i$ lie on four distinct vertices $\{u_i, v_i, x_i, y_i\}$ with red spine $e_i = u_i v_i \in N$) and $b - c$ one-page books $\mathcal{B}_i = \{T_i\}$ ($i \in I_1 := B^* \setminus I_2$). Set $W := B(I_1) \subseteq O'$, so $|W| = 2(b - c)$.

Random-bipartition cover and outside-N red edges. For a uniform coloring $\chi : V \to \{0,1\}$ over all $2^{|V|} \ge 1$ assignments, call uv *non-crossing* if $\chi(u) = \chi(v)$. Select $S_{\mathrm{blue}}(\chi) := W \cup \{e \in O' \setminus W : \chi \text{ non-crossing on } e\}$ (`selectedOld`) and add all non-old edges of triangles of G' missed by $S_{\mathrm{blue}}(\chi)$:

$$D_{\mathrm{red}}(\chi) := \bigcup_{\substack{T \in \mathcal{T}(G') \\ T \cap S_{\mathrm{blue}}(\chi) = \emptyset}} (T \setminus O')$$

(`cutAddedEdges`). By pigeonhole on $\{0,1\}$, every triangle of G' has a non-crossing edge (`IsTriangle.exists_noncrossing`); hence every type-3 triangle meets $S_{\mathrm{blue}}(\chi)$, every missed triangle belongs to $\mathcal{F}_2$ and contributes its non-crossing red edge to $D_{\mathrm{red}}(\chi)$, and $E(A) \cup S_{\mathrm{blue}}(\chi) \cup D_{\mathrm{red}}(\chi)$ covers $\mathcal{T}(G)$ (`cutCover_covers`). Averaging over χ (`selectedOld_average`) gives

$$\mathbb{E}_\chi\big[|S_{\mathrm{blue}}(\chi)|\big] = |W| + \frac{1}{2}\big(|O'| - |W|\big) = \frac{3}{2}n - \frac{1}{2}a + b - c. \tag{3.1}$$

Any added red edge $f \in D_{\mathrm{red}}(\chi) \setminus N$ comes from a triangle $T \in \mathcal{F}_2$ missing $W \cup N$, which by Lemma 2.2's book witness (`cutAddedEdges_book_witness`) satisfies $T \in \{u_i x_i y_i, v_i x_i y_i\}$ for some two-page book $i \in \mathcal{B}_{\mathrm{out}} \subseteq I_2$ (`helperRoots`) whose opposite pair $f = x_i y_i$ is a red edge of G' outside N. By Lemma 2.1, $\hat{h} := |\mathcal{B}_{\mathrm{out}}| \le h$, and for each $i \in \mathcal{B}_{\mathrm{out}}$ the event that $\chi(x_i) = \chi(y_i)$ while $\{u_i, v_i\}$ are not both colored $\chi(x_i)$ has probability $\frac{1}{2} - \frac{1}{8} = \frac{3}{8}$ (`bookEvent_average`), giving

$$\mathbb{E}_\chi\big[|D_{\mathrm{red}}(\chi) \setminus N|\big] \le \frac{3}{8}\hat{h}. \tag{3.2}$$

Helper functional digraph and disjoint sink-arborescences. Define helperNext : $B^* \to \mathrm{Option}\, B^*$ by setting $\mathrm{helperNext}(j) = \mathrm{some}\, i$ ($j \to i$) iff $j \in I_2$ and its opposite pair $\{x_j, y_j\}$ equals the spine $e_i = \{u_i, v_i\}$ of $i \in B^*$. Spine injectivity (`spine_injective`) makes helperNext well-defined, and $x_r y_r \notin N$ gives $\mathrm{helperNext}(r) = \mathrm{none}$ for all $r \in \mathcal{B}_{\mathrm{out}}$ (`helperNext_eq_none_of_roots`).

Lemma 3.1 (Sink-rooted arborescence structure; `Probability/Arborescence.lean`)**.** *Let* $\mathrm{next} : I \to \mathrm{Option}\, I$ *be a partial function on a finite set* I *and* $R_0 \subseteq I$ *a set of sinks* ($\mathrm{next}(r) = \mathrm{none}$)*. For* $r \in R_0$ *and* $v \in I$*, set* $\mathcal{T}_r := \{v \in I : \exists k \in \mathbb{N},\ \mathrm{reachIn}(\mathrm{next}, k, v) = \mathrm{some}\, r\}$ *and* $\mathrm{children}(v) := \{j \in I : \mathrm{next}(j) = \mathrm{some}\, v\}$*, with* $\deg^-(v) := |\mathrm{children}(v)|$.

(a) If $\mathrm{reachIn}(\mathrm{next}, k_1, v) = \mathrm{some}\, r_1$ *and* $\mathrm{reachIn}(\mathrm{next}, k_2, v) = \mathrm{some}\, r_2$ *for* $r_1, r_2 \in R_0$*, then* $k_1 = k_2$ *and* $r_1 = r_2$ (***reachIn_sink_unique***)*, so every* $v \in \mathcal{T}_r$ *has a unique cycle-free directed path* $v = i_k \to i_{k-1} \to \cdots \to i_0 = r$ *in* $\mathcal{T}_r$ (***exists_chain_from_sink***)*.*

(b) The basins $\{\mathcal{T}_r : r \in R_0\}$ *are pairwise disjoint* (***basin_disjoint***)*, each has* $|\mathcal{T}_r| \geq 1$ *and* $\bigsqcup_{v \in \mathcal{T}_r} \mathrm{children}(v) = \mathcal{T}_r \setminus \{r\}$*, and* $\sum_{v \in \mathcal{T}_r} \deg^-(v) + 1 = |\mathcal{T}_r|$ (***sum_childrenOf_card***)*.*

(c) For every $k \in \mathbb{N}$*,*

$$\frac{1}{8} - \frac{3}{32}k \leq \frac{1}{8} \cdot 4^{-k} \tag{3.3}$$

with equality at $k \in \{0, 1\}$ (***fourth_pow_secant***)*. Hence if* $p : I \to \mathbb{Q}$ *obeys* $p(i) \leq \frac{1}{2}$ *on* I *and* $p(v) \leq \frac{1}{2} - \frac{1}{8}4^{-\deg^-(v)}$ *on* $\bigsqcup_{r \in R_0} \mathcal{T}_r$*, then* $\sum_{v \in \mathcal{T}_r} p(v) \leq \frac{1}{2}|\mathcal{T}_r| - \frac{1}{8}$ (***basin_prob_sum_le***) *and* $\sum_{i \in I} p(i) \leq \frac{1}{2}|I| - \frac{1}{8}|R_0|$ (***global_prob_sum_le***)*.*

Proof. (a) Induct on k_1: at $k_1 = 0$, $v = r_1$ has $\mathrm{next}(v) = \mathrm{none}$, forcing $k_2 = 0$ and $r_1 = r_2$; at $k_1 = m_1 + 1$, $\mathrm{next}(v) = \mathrm{some}\, u$ rules out $k_2 = 0$ and passes to u. Step-count uniqueness forces all vertices along $v \to \cdots \to r$ to be distinct (directed cycles can exist in $I \setminus \bigsqcup_{r \in R_0} \mathcal{T}_r$, as in Table 2, but cannot reach any sink).

(b) Sink uniqueness gives $\mathcal{T}_{r_1} \cap \mathcal{T}_{r_2} = \emptyset$ for $r_1 \neq r_2$. Since $r \in \mathcal{T}_r$ and $j \in \mathcal{T}_r \setminus \{r\}$ iff $\mathrm{next}(j) = \mathrm{some}\, v$ for a unique $v \in \mathcal{T}_r$, summing child counts gives $\sum_{v \in \mathcal{T}_r} \deg^-(v) + 1 = |\mathcal{T}_r \setminus \{r\}| + 1 = |\mathcal{T}_r|$.

(c) Inequality (3.3) is an equality at $k = 0$ ($\frac{1}{8}$) and $k = 1$ ($\frac{1}{32}$), while for $k \geq 2$ its left side is $\leq -\frac{1}{16} < 0 < \frac{1}{8}4^{-k}$. For $s_r := |\mathcal{T}_r| \geq 1$, summing (3.3) with (b) gives

$$\sum_{v \in \mathcal{T}_r} p(v) \leq \frac{3}{8}s_r + \frac{3}{32}(s_r - 1) = \frac{1}{2}s_r - \Big(\frac{3}{32} + \frac{1}{32}s_r\Big) \leq \frac{1}{2}s_r - \frac{1}{8},$$

and bounding $p(i) \leq \frac{1}{2}$ on $I \setminus \bigsqcup_{r \in R_0} \mathcal{T}_r$ yields $\sum_{i \in I} p(i) \leq \frac{1}{2}|I| - \frac{1}{8}|R_0|$. □

Lemma 3.2 (All-depth 2^r binary-toggle exchange; `basin_third_vertex_child`)**.** *For every* $r_0 \in \mathcal{B}_{\mathrm{out}}$ *and* $v \in \mathcal{T}_{r_0}$ *(so* $v \in I_2$ *is a two-page book* $\{u_v v_v x_v, u_v v_v y_v\}$*), define*

$$S_v := \big\{z \in V \setminus \{u_v, v_v\} : u_v v_v z \in \mathcal{F}_2 \text{ and } u_v z \notin W,\ v_v z \notin W\big\}.$$

Then every $z \in S_v \setminus \{x_v, y_v\}$ *belongs to* $e_j = \{u_j, v_j\}$ *for some child* $j \in \mathrm{children}(v)$ ($j \to v$)*, whence* $|S_v| \leq 2 + 2\deg^-(v)$.

Proof. Let $v = i_r \to i_{r-1} \to \cdots \to i_1 = r_0$ ($r \geq 1$) be the unique chain of distinct two-page books from v to r_0, with $\{x_{i_s}, y_{i_s}\} = e_{i_{s-1}}$ for $2 \leq s \leq r$ and $f_1 := x_{i_1} y_{i_1} \in (E(G') \setminus O) \setminus N$.

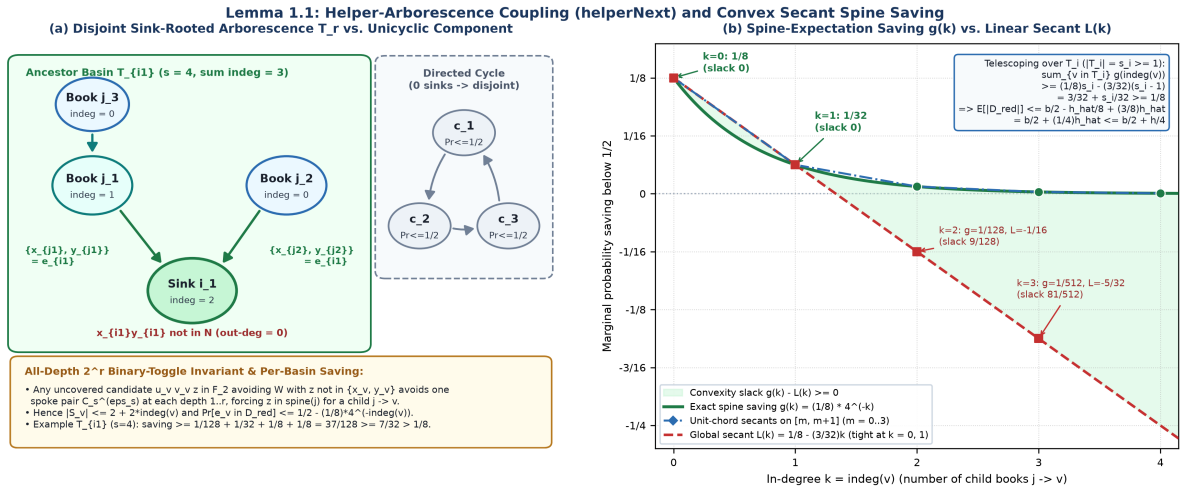


Figure 2: Mechanism of Lemma 1.1. Panel (a) depicts a four-book ancestor basin $\mathcal{T}_{i_1}$ ($s = 4$, $\sum \deg^- = 3$) rooted at a sink $i_1 \in \mathcal{B}_{\text{out}}$ alongside a disjoint directed 3-cycle and the 2^r binary-toggle bound $|S_v| \le 2 + 2\deg^-(v)$. Panel (b) compares the exact marginal spine saving $g(k) = \frac{1}{8}4^{-k}$ against the global secant $L(k) = \frac{1}{8} - \frac{3}{32}k$ and unit-chord secants.

For each $1 \le s \le r$, define the candidate pair $C_s^{(0)} := x_{i_s}y_{i_s}u_{i_s}$ and $C_s^{(1)} := x_{i_s}y_{i_s}v_{i_s}$ in $\mathcal{F}_2$ (`Book.cand`). Their red edges $f_1, e_{i_1}, \dots, e_{i_{r-1}}$ are pairwise distinct and distinct from $e_{i_r} = e_v$, and their blue spoke pairs $B(C_s^{(0)}) = \{u_{i_s}x_{i_s}, u_{i_s}y_{i_s}\}$ and $B(C_s^{(1)}) = \{v_{i_s}x_{i_s}, v_{i_s}y_{i_s}\}$ partition $B(T_{i_s}) \sqcup B(Q^*_{i_s})$ into two disjoint 2-sets (`spokes_disjoint`, `cand_properties`).

Given $z \in S_v \setminus \{x_v, y_v\}$, put $Z := u_{i_r}v_{i_r}z \in \mathcal{F}_2$, so $B(Z) = \{u_{i_r}z, v_{i_r}z\} \subseteq O' \setminus W$ (`third_vertex_spokes_oldRemaining`). For each $1 \le s \le r$, $B(Z)$ cannot meet both $B(C_s^{(0)})$ and $B(C_s^{(1)})$ (`exists_disjoint_spokes`): since the two spoke pairs are disjoint on the four distinct vertices $\{u_{i_s}, v_{i_s}, x_{i_s}, y_{i_s}\}$ and the two edges of $B(Z)$ share z, meeting both forces $z \in \{x_{i_s}, y_{i_s}\}$ and $\{u_{i_r}, v_{i_r}\} = \{u_{i_s}, v_{i_s}\}$ (`pair_spokes_forced`), contradicting $z \notin \{x_v, y_v\}$ if $s = r$ and contradicting $e_{i_r} \ne e_{i_s}$ if $s < r$ (with zero cross-book vertex-disjointness assumptions).

Choosing $\varepsilon_s \in \{0, 1\}$ with $B(Z) \cap B(C_s^{(\varepsilon_s)}) = \emptyset$ for $1 \le s \le r$ yields a packing $\mathcal{C} := \{C_1^{(\varepsilon_1)}, \dots, C_r^{(\varepsilon_r)}, Z\}$ of $r + 1$ triangles in $\mathcal{F}_2$. Let $\mathcal{M}_{\text{ext}} := \{j \in B^* \setminus \{i_1, \dots, i_r\} : B(Z) \cap E(T_j) \ne \emptyset\} \subseteq I_2 \setminus \{i_1, \dots, i_r\}$ (since $B(Z) \cap W = \emptyset$). Because $|B(Z)| = 2$:

- if $|\mathcal{M}_{\text{ext}}| = 0$, then $(B^* \setminus \{T_{i_1}, \dots, T_{i_r}\}) \cup \mathcal{C}$ packs $b + 1$ triangles in $\mathcal{F}_2$, contradiction;
- if $|\mathcal{M}_{\text{ext}}| = 2$ ($\{j_1, j_2\}$), then $B(Z) \subseteq B(B^*)$ avoids $B(Q)$, so $(B^* \setminus \{T_{i_1}, \dots, T_{i_r}, T_{j_1}, T_{j_2}\}) \cup \mathcal{C} \cup \{Q^*_{j_1}, Q^*_{j_2}\}$ has size $b + 1$ in $\mathcal{F}_2$, contradiction;
- thus $\mathcal{M}_{\text{ext}} = \{j\}$ for a unique $j \in I_2 \setminus \{i_1, \dots, i_r\}$, and $B(Z)$ must also meet $B(Q^*_j)$ (else replacing $\{T_{i_1}, \dots, T_{i_r}, T_j\}$ by $\mathcal{C} \cup \{Q^*_j\}$ gives $b + 1$).

Hence Z meets both pages of $\mathcal{B}_j$ while avoiding e_j ($j \ne i_r$). By `two_page_triangle`, $Z \in \{u_jx_jy_j, v_jx_jy_j\}$; matching unique red edges gives $x_jy_j = e_{i_r} = e_v$ ($j \to v$) and $z \in \{u_j, v_j\}$. Thus $S_v \subseteq \{x_v, y_v\} \cup \bigcup_{j \to v} e_j$ and $|S_v| \le 2 + 2\deg^-(v)$. □

Proof of Lemma 1.1. Set $p(i) := \mathbb{P}_\chi[e_i \in D_{\text{red}}(\chi)] \le \frac{1}{2}$ for $i \in B^*$. For $r_0 \in \mathcal{B}_{\text{out}}$ and

$v \in \mathcal{T}_{r_0}$, $e_v = u_v v_v \in D_{\mathrm{red}}(\chi)$ implies $\chi(u_v) = \chi(v_v)$ and $\chi(z) \neq \chi(u_v)$ for some $z \in S_v$. Since $\{u_v, v_v\} \sqcup S_v$ are $|S_v| + 2$ distinct vertices with independent bits, Lemma 3.2 gives (`spineEvent_average`, `spine_prob_le_in_basin`)

$$p(v) \le \frac{1}{2} - \left(\frac{1}{2}\right)^{|S_v|+1} \le \frac{1}{2} - \frac{1}{8} \cdot 4^{-\deg^-(v)}.$$

By linearity of expectation (requiring zero independence across coupled edges $j \to v$), Lemma 3.1(c) with $I = B^*$ and $R_0 = \mathcal{B}_{\mathrm{out}}$ ($\hat{h} \le h$) together with (3.2) (Figure 2) gives

$$\mathbb{E}_\chi\big[|D_{\mathrm{red}}(\chi)|\big] \le \Big(\frac{b}{2} - \frac{1}{8}\hat{h}\Big) + \frac{3}{8}\hat{h} = \frac{b}{2} + \frac{1}{4}\hat{h} \le \frac{b}{2} + \frac{1}{4}h.$$

Adding $|E(A)| = 3a$ and (3.1) bounds the expected global cover size by $\frac{3}{2}n + \frac{5}{2}a + \frac{3}{2}b - c + \frac{1}{4}\hat{h}$ before relaxing $\hat{h} \le h$, so finite averaging (`exists_le_finiteAverage`) establishes (4″) (`Construction.helper_arborescence_random_cover_unrelaxed`) and hence (4′) (`Construction.helper_arborescence_random_cover`). □

Remark 3.3 (Intermediate Depth-1 sub-bounds). *(Verified in* ***`recompute.py`****; not formalized in* ***`lean/TuzaBound`****.)* Stopping Lemma 3.2 at depth $r = 1$ already beats $\frac{165}{59}$: if $j \to i \in \mathcal{B}_{\mathrm{out}}$, then $j \notin \mathcal{B}_{\mathrm{out}}$ and j cannot carry a third page $Z \in Q_1 \subseteq \mathcal{S}_Q$ (applying Lemma 3.2's $r = 2$ exchange along $j \to i$ to Z forces $B(Z) \cap B(B^*) \neq \emptyset$), so at most $c - \hat{h} - d$ arcs enter $\mathcal{B}_{\mathrm{out}}$ and at least $\max(0, 2\hat{h} - c + d)$ roots have $\deg^-(i) = 0$. This yields

$$\tau(G) \le \frac{3}{2}n + \frac{5}{2}a + \frac{3}{2}b - \frac{7}{8}c - \frac{1}{8}d + \frac{1}{8}h,$$

giving $1649\,\tau(G) + 94\,h_1 \le 4611\,\nu(G)$ ($\frac{4611}{1649} \approx 2.796240$) via integer weights $(560, 200, 24, 529, 78, 258)$, and $\tau(G) \le \frac{652959}{233525}\,\nu(G) \approx 2.796099\,\nu(G)$ when adjoined to unit-chord secants of $\frac{1}{8}4^{-k}$ on $[m, m+1]$ (active on $[2,3]$ and $[3,4]$).

3.2 Unmerged colored covers and proof of Theorem 1

Proof of Lemma 1.2. On the fixed tuple (P, A, P', Q, Q_1), every triangle of G missing $E(A) \cup B(P')$ belongs to $\mathcal{S}$ (Section 2.3(4)), and $|E(A)| + |B(P')| = 3a + 3m - b \le 3n - b$, so $\tau(G) + b \le 3n + \tau(\mathcal{S})$ (`exists_lifted_special_cover`, `transversalNumber_le_add_specialFamily`). Applying Lemma 2.3(i), the elementary cover $\tau(\mathcal{S}) + h_1 \le 2c + 3d$ of Lemma 2.3(ii) (`second_colored_cover`), and Lemma 2.3(iii) to $(G' - B(P'), Q, Q_1)$ on the exact same (Q, Q_1) yields (6a) (`unmerged_cover_6a`), (6b) (`unmerged_cover_6b`), and (6c) (`unmerged_cover_6c`). □

Proof of Theorem 1. Combine (3), (4′), (5), (6a), and (6c) (`Construction.five_covers`) on (P, A, P', Q, Q_1) with integer weights $(170, 17, 160, 24, 26) = 2 \times (85, \frac{34}{4}, 80, 12, \frac{39}{3})$ (`five_bounds_arithmetic`), writing $t := \tau(G)$:

$$\begin{aligned} 170\,(t + a) &\le 510\,n, \qquad 160\,(t + c + d) \le 480\,n, \\ 17\,(4t + 4c) &\le 102\,n + 170\,a + 102\,b + 17\,h, \\ 24\,(t + b + 2d + h) &\le 72\,n + 72\,c + 24\,h_1, \qquad 26\,(3t + 3b + 2h_1) \le 234\,n + 156\,c + 208\,d. \end{aligned}$$

Summing gives $(170 + 68 + 160 + 24 + 78)t = 500\,t$ on the left, $(510 + 102 + 480 + 72 + 234)n = 1398\,n$ on the right, exact cancellation on a ($-170 + 170 = 0$), b ($102 - 24 - 78 = 0$),

c ($-68 - 160 + 72 + 156 = 0$), and d ($-160 - 48 + 208 = 0$), and net left-hand slack $(24-17)h + (52-24)h_1 = 7h + 28h_1 \geq 0$. This proves (1.3) (`Construction.bound_with_private_slack`), and dividing $500\,t \leq 1398\,\nu(G)$ by 2 gives (1.2) (`tuza_bound_nat`, `tuza_bound`). □

Corollary 3.4 (Recovery of Wang's bound; `wang_bound_from_699_250`). *Every finite simple graph G satisfies $59\,\tau(G) \leq 165\,\nu(G)$.*

Proof. Multiply $250\,\tau(G) \leq 699\,\nu(G)$ by 59 and use $59 \times 699 = 41241 \leq 41250 = 165 \times 250$. □

3.3 Wang's remark audit, infinite-depth LP fixed point, and the $\mathcal{S}$-layer gadget $H_{14,3,1}$

Proposition 3.5 (Audit of Wang's equality-point remark; verified in `recompute.py`, not in `lean/TuzaBound`). *At $(a,m,b,c,d,h) = (\frac{12}{59}, \frac{39}{59}, \frac{39}{59}, \frac{12}{59}, 0, 0)n$ ([18, p. 5]), $a+m = \frac{51}{59}n < n$, so the unrelaxed third cover in (5) has size $3a+3m-c-d = \frac{141}{59}n < \frac{165}{59}n$. Even after raising m to $\frac{47}{59}n$, $d = 0$ implies $\mathcal{S}_Q = \emptyset$, so $E(A) \cup B(P') \cup B(Q)$ covers $\mathcal{T}(G)$ with $3a+3m-b+2c = \frac{162}{59}n < \frac{165}{59}n$. Adjoining that $d = 0$ cover forces Wang's tight LP ray onto $(a,m,b,c,d,h) = (\frac{12}{59}, \frac{47}{59}, \frac{36}{59}, \frac{12}{59}, 0, \frac{12}{59})n$, where $h = c > 0$.*

Proof. Direct substitution gives $\frac{36+117-12}{59}n = \frac{141}{59}n$ at $m = \frac{39}{59}n$ and $\frac{36+102+24}{59}n = \frac{162}{59}n$ at $m = \frac{47}{59}n$. Since Wang's weights $(20,8,19,12)$ give $59\,\tau + d \leq 165\,n$, achieving $\tau \geq \frac{165}{59}n$ forces $d = 0$; requiring $3n-a \geq \frac{165}{59}n$, $3n-c \geq \frac{165}{59}n$, $a+m \leq n$, and $3a+3m-b+2c \geq \frac{165}{59}n$ then gives $\frac{12}{59}n - a \geq 0$, $\frac{12}{59}n - c \geq 0$, $\frac{12}{59}n + 2c - b \geq 3(n-a-m) \geq 0$, and $c - h \geq 0$. By the exact algebraic identity

$$\frac{3}{2}n + \frac{5}{2}a + \frac{3}{2}b - c + \frac{3}{8}h = \frac{165}{59}n - \frac{5}{2}\Big(\frac{12}{59}n - a\Big) - \frac{3}{2}\Big(\frac{12}{59}n + 2c - b\Big) - \frac{19}{8}\Big(\frac{12}{59}n - c\Big) - \frac{3}{8}(c-h),$$

requiring Wang's (4) to be at least $\frac{165}{59}n$ forces all four non-negative slacks (and hence $n-a-m$) to vanish, so $a = c = h = \frac{12}{59}n$, $b = \frac{12}{59}n + 2c = \frac{36}{59}n$, and $m = \frac{47}{59}n$. □

Proof of Theorem 2(a). Following `lean/TuzaBound/FixedPoint.lean`:

1. *Two-page dominance (`book2_le_book3`):* Since $p_{r+2} + u_{r+1} \leq p_{r+1}$, we have $-2p_{r+1} + u_{r+1} \leq -p_{r+1} - p_{r+2}$, so $3p_r - 2p_{r+1} + u_{r+1} - u_r \leq 3p_r - p_{r+1} - p_{r+2} - u_r$ for every $r \geq 0$.
2. *Exact $(\frac{8}{3}, \frac{2}{3})$ fixed point (`fixed_point_recurrence_eq`, `fixed_point_recurrence`):* At convex weight $\lambda = \frac{1}{3}$,

$$\begin{aligned}\frac{2}{3}\big(3p_r - 2p_{r+1} + u_{r+1} - u_r\big) + \frac{1}{3}\Big(2p_r + \frac{8}{3}p_{r+1} - \frac{2}{3}u_{r+1}\Big) \\ = \frac{8}{3}p_r - \frac{2}{3}u_r - \frac{4}{9}(p_{r+1} - u_{r+1}),\end{aligned}$$

   which is at most $\frac{8}{3}p_r - \frac{2}{3}u_r \leq 3p_r - u_r$ since $u_{r+1} \leq p_{r+1}$ and $u_r \leq p_r$.

Thus every feasible point of the depth-K ($K \geq 1$) or infinite-depth LP satisfies (3), (4′), (5), (6a) (book2_0), and (6c) (col$_1$) with $(c,d,h,h_1) = (p_0, p_1, u_0, u_1)$, so weights $(85,34,80,12,39)$ give $250\,\tau \leq 699\,n$ (`infinite_depth_lp_upper`). Conversely, at

$(n, \tau, a, m, b) = (250, 699, 51, 199, 159)$, $p = (42, 9, 3, 0, \dots)$, and $u = (0, 0, 3, 0, \dots)$ (`infinite_depth_lp_attained`), $a + m = 250$, the disjoint blue-edge budget is $51 + 438 + 108 = 597 \le 750$, (3), (4′), and (5) all equal 699, and with base $3a + 3m - b = 591$ and $\mathrm{bluePrefix}(p, r) = (0, 84, 102, 108, \dots)$, the four depth-$r$ bounds $(\mathrm{triv}_r, \mathrm{col}_r, \mathrm{book2}_r, \mathrm{book3}_r)$ evaluate to $(717, 703, 699, 705)$ at $r = 0$, $(702, 699, 699, 699)$ at $r = 1$, and $(699, 699, 699, 699)$ for all $r \ge 2$, proving $\mathrm{OPT}_K = \frac{699}{250}$ (`infinite_depth_lp_exact_699_250`). □

Proof of Theorem 2(b). At $n = 250$, $(p_0, p_1, p_2) = 3 \times (14, 3, 1)$, $(u_0, u_1, u_2) = 3 \times (0, 0, 1)$, and $699 - 591 = 3 \times 36$. Construct $H_{14,3,1}$ on one block $(14, 3, 1)$ as follows:

- *Spine 0:* Five vertices $u_0, v_0, y_{0,0}, y_{1,0}, y_{2,0}$, red edge u_0v_0, and six blue edges $\{u_0y_{s,0}, v_0y_{s,0} : s \in \{0, 1, 2\}\}$.
- *Spines 1, 2:* Set $u_i := y_{1,0}$, add fresh $v_i, y_{0,i}, y_{1,i}$, red edge u_iv_i, four blue edges $\{u_iy_{0,i}, v_iy_{0,i}, u_iy_{1,i}, v_iy_{1,i}\}$, and cross blue edge v_iu_0 (yielding witness $u_0y_{1,0}v_i \in \mathcal{F}_1$ using $u_0y_{1,0} \in B(Q_{1,0})$).
- *Spine 3 (opposite-edge identification):* Set $u_3 := y_{1,0}$, $v_3 := y_{2,0}$, add fresh $y_{0,3}$, red edge $y_{1,0}y_{2,0}$, and two blue edges $\{y_{1,0}y_{0,3}, y_{2,0}y_{0,3}\}$. Identifying spine 3 with the opposite pair $y_{1,0}y_{2,0}$ of $(Q_{1,0}, Q_{2,0})$ keeps $|R(Q_0)| = 14$ while supplying witnesses $u_0y_{1,0}y_{2,0}, v_0y_{1,0}y_{2,0} \in \mathcal{F}_1$ (both hit $B(Q_{1,0})$) and forcing Lemma 2.3(i) on (Q_1, Q_2) to pay $+1$ for $y_{1,0}y_{2,0} \notin R(Q_1)$.
- *Tail spines $4, \dots, 13$:* Set $u_i := y_{0,0}$, add fresh $v_i, y_{0,i}$, red edge u_iv_i, blue edges $\{u_iy_{0,i}, v_iy_{0,i}\}$, and cross blue edge v_iu_0 (yielding witness $Q^{**}_{0,i} := y_{0,0}v_iu_0 \in \mathcal{F}_0$, which hits $u_0y_{0,0} \in B(Q_{0,0})$).

Direct enumeration over $\binom{32}{3} = 4960$ triples (`recompute.py`) confirms 32 vertices, 48 blue and 14 red edges (62 total), 32 triangles (all in $\mathcal{F}_0$), $\nu(\mathcal{F}_0) = 14$ via $Q_0 := \{u_iv_iy_{0,i}\}_{i=0}^{13}$ (since $|E_{\mathrm{red}}| = 14$ covers all 32 triangles, global $\nu = \tau^* = \tau = 14$), $\nu(\mathcal{F}_1) = 3$ via $Q_1 := \{Q_{1,0}, Q_{1,1}, Q_{1,2}\}$, $\mathcal{F}_2 = \{Q_{2,0}\}$ ($p_2 = u_2 = 1$, $u_0 = u_1 = 0$), and exact size 36 on all three Lemma 2.3 $\mathcal{S}$-covers:

$$3p_0 - 2p_1 + u_1 - u_0 = 42 - 6 = 36, \qquad 2p_0 + (3p_1 - 2p_2 + u_2 - u_1) = 28 + 8 = 36,$$

and $2p_0 + 2p_1 + (3p_2 - u_2) = 34 + 2 = 36$.

For the two $n = 250$ whole-graph obstructions:

(i) Every $e_v \in N$ obeys $\mathbb{P}_\chi[e_v \in D_{\mathrm{red}}] = \frac{1}{2}(1 - 2^{-|S_v|}) < \frac{1}{2}$ on any finite graph; at $(n, a, b, c, h) = (250, 51, 159, 42, 0)$, if all $b - c = 117$ one-page spines have $S_e = \emptyset$, then

$$\mathbb{E}_\chi\big[|C^{(4')}|\big] \le 3(51) + 2(117) + \frac{699 - 234}{2} + \frac{42}{2} = \frac{1281}{2} = 640.5 < 699.$$

(ii) Suppose three disjoint copies of $H_{14,3,1}$ form $\mathcal{S}$ at $(n, a, m, b, c, d) = (250, 51, 199, 159, 42, 9)$. For any tail spine $i \in \{4, \dots, 13\}$ ($T_i = u_iv_ix_i \in B_2^*$, $Q_{0,i} = u_iv_iy_{0,i} \in Q_0$), $y_{0,i}$ is non-incident to $Q^{**}_{0,i} = y_{0,0}v_iu_0$, so any triangle $U_i \in \{u_ix_iy_{0,i}, v_ix_iy_{0,i}\}$ on an opposite edge $x_iy_{0,i} \in E(G') \setminus C^{(5)}_{\mathrm{base}}$ is edge-disjoint from $Q^{**}_{0,i}$. If $x_iy_{0,i} \notin O'$ is red, then $x_iy_{0,i} \notin N$ ($N \subseteq C^{(5)}_{\mathrm{base}}$), so Lemma 2.1 contradicts $Q^{**}_{0,i} \in \mathcal{S}$. If $x_iy_{0,i} \in O' \setminus B(P')$ is blue, then $(P' \setminus \{T_i\}) \cup \{Q^{**}_{0,i}, U_i\}$ ($Q^{**}_{0,i} \in \mathcal{F}_2$, $U_i \in \mathcal{F}_3$) has counts $(b, m + 1)$, contradicting lexicographic maximality of P'. If $x_iy_{0,i} \in B(T_k)$ for $T_k \in B_2^*$ ($k \ne i$, since $B(P' \setminus B_2^*) \subseteq C^{(5)}_{\mathrm{base}}$), then $Q^{**}_{0,i}$ and $B(Q_{0,i})$ are disjoint from $Q_{0,k}$ ($k \ne 0$) and from $Q_{1,0}$ ($k = 0$), so replacing $\{T_i, T_k\} \subseteq P'$ by $\{Q^{**}_{0,i}, Q_{0,k}, U_i\}$ ($k \ne 0$) or $\{Q^{**}_{0,i}, Q_{1,0},$

$U_i\}$ ($k = 0$) again yields $(b, m+1)$, contradiction. Deleting these $3 \times 10 = 30$ missing opposite edges caps $|C^{(5)}| \le 699 - 30 = 669 < 699$.

□

3.4 All-graph opposite-edge elimination and proof of Theorem 3

We now promote Theorem 2(b)(ii)'s exchange obstruction to every finite simple graph, following `Construction/OppositeElimination.lean` and `Main.lean`. On (P, A, P', Q), index the c two-page books of (B^*, Q) as $\mathcal{B}_i = \{T_i = u_i v_i x_i, Q_i^* = u_i v_i y_i\}$ ($i \in I_2$, $|I_2| = c$) with red spine $e_i = u_i v_i \in N$ and opposite pair $f_i := \{x_i, y_i\}$. Partition $I_2 = I_{\mathrm{np}} \sqcup I_{\mathrm{priv}}$, where I_{priv} ($|I_{\mathrm{priv}}| = h$, `privPages`) indexes spines private in $\mathcal{S}$ and $|I_{\mathrm{np}}| = c - h$. By Lemma 2.1 (`rootPages_subset_privPages`), every $k \in \mathcal{B}_{\mathrm{out}}$ ($f_k \in (E(G') \setminus O') \setminus N$) belongs to I_{priv}, giving $I_{\mathrm{priv}} = \mathcal{B}_{\mathrm{out}} \sqcup I_{\mathrm{priv},0}$ (`privZeroPages`) with $\hat{h} + h_0 = h$ (`helperRoots_add_privZeroPages_card`).

Proof of Lemma 1.3. Put $I_{\mathrm{act}} := I_{\mathrm{np}} \sqcup \mathcal{B}_{\mathrm{out}}$ ($|I_{\mathrm{act}}| = c - h + \hat{h} = c - h_0$, `activePages`) and define

$$C_{\mathrm{core}} := E(A) \ \cup \ E\big(P' \setminus \{T_i : i \in I_{\mathrm{act}}\}\big) \ \cup \ \{e_i : i \in I_{\mathrm{act}}\} \ \cup \ \{f_k : k \in \mathcal{B}_{\mathrm{out}}\}.$$

Counting sizes gives $|C_{\mathrm{core}}| \le 3a + 3(m - c + h_0) + (c - h_0) + \hat{h} = 3a + 3m - 2c + \hat{h} + 2h_0 \le 3n - 2c + \hat{h} + 2h_0$. To see that C_{core} covers $\mathcal{T}(G)$ (`remainingCoreCover_covers`), apply Lemma 2.2 in G' to the m pairwise edge-disjoint books consisting of $(m - c + h_0)$ single-page books $\{T\}$ ($T \in P' \setminus \{T_i : i \in I_{\mathrm{act}}\}$) and $(c - h_0)$ two-page books $\{T_i, Q_i^*\}$ ($i \in I_{\mathrm{act}}$): every page choice has b members in $\mathcal{F}_2$ and m total members, hence is maximal in $\mathcal{T}(G')$ by lexicographic maximality of P'. Its base set $C_0 \subseteq C_{\mathrm{core}} \setminus E(A)$ contains N, $W = B(B_1^*)$, $B(P' \setminus B^*)$, and $\bigsqcup_{k \in I_{\mathrm{priv},0}} B(T_k)$.

Suppose some $T \in \mathcal{T}(G)$ avoids C_{core}. Then $T \in \mathcal{T}(G')$ avoids C_0, so Lemma 2.2 yields $i \in I_{\mathrm{act}}$ with $T \in \{U_i^{(u)} := u_i x_i y_i, U_i^{(v)} := v_i x_i y_i\}$ and $f_i = x_i y_i \in E(G') \setminus C_{\mathrm{core}}$:

1. *Step 1 ($i \in I_{\mathrm{np}}$ and $f_i \in O'$ is blue):* Since $\{f_k : k \in \mathcal{B}_{\mathrm{out}}\} \subseteq C_{\mathrm{core}}$, we have $i \notin \mathcal{B}_{\mathrm{out}}$ and hence $i \in I_{\mathrm{np}}$, yielding a second page $Z_i = u_i v_i z_i \in \mathcal{S} \setminus \{Q_i^*\}$ on five distinct vertices $\{u_i, v_i, x_i, y_i, z_i\}$. If $f_i \notin O'$ were red, then $f_i \notin N$ ($N \subseteq C_0$), so Lemma 2.1 would contradict $i \in I_{\mathrm{np}}$. Thus $f_i \in O' \setminus C_{\mathrm{core}}$ is blue and $U_i^{(u)}, U_i^{(v)} \in \mathcal{F}_3$.
2. *Step 2 ($f_i \notin O' \setminus B(P')$; `IsLexMaximumPacking.no_one_to_two`):* If $f_i \in O' \setminus B(P')$, then $V(U_i^{(u)}) \cap V(Z_i) = \{u_i\}$ makes $U_i^{(u)} \in \mathcal{F}_3$ and $Z_i \in \mathcal{F}_2$ edge-disjoint and disjoint from $E(P' \setminus \{T_i\})$, giving a $(b, m+1)$ packing $(P' \setminus \{T_i\}) \cup \{Z_i, U_i^{(u)}\}$, contradiction. Hence $f_i \in B(T_k)$ for some $k \in I_{\mathrm{act}} \setminus \{i\}$.
3. *Step 3 ($f_i \notin B(T_k)$ for $k \in I_{\mathrm{np}} \setminus \{i\}$; `step3_nonprivate_obstruction`):* Let $Z_k = u_k v_k z_k \in \mathcal{S} \setminus \{Q_k^*\}$ on $\{u_k, v_k, y_k, z_k\}$ ($B(Q_k^*) \cap B(Z_k) = \emptyset$). Because $B(Q_i^*) \cap B(Q_k^*) = \emptyset$, $U_i^{(u)}$ is edge-disjoint from Q_k^*, so $B(Z_i) \cap B(Q_k^*) \neq \emptyset$ (else replacing $\{T_i, T_k\}$ by $\{U_i^{(u)}, Z_i, Q_k^*\}$ gives $(b, m+1)$, `no_two_to_three`). Likewise $u_i y_i$ and $v_i y_i$ cannot both lie in $B(Z_k)$ (else $e_i = e_k$), so choosing $\alpha \in \{u_i, v_i\}$ with $\alpha y_i \notin B(Z_k)$ makes $U_i^{(\alpha)}$ disjoint from Z_k and forces $B(Z_i) \cap B(Z_k) \neq \emptyset$. Thus $B(Z_i) = \{u_i z_i, v_i z_i\}$ picks $g_1 \in B(Q_k^*)$ and $g_2 \in B(Z_k)$ sharing z_i, forcing $z_i \in \{u_k, v_k\}$ and $\{u_i, v_i\} = \{y_k, z_k\}$,

so $y_k \in \{u_i, v_i\}$. Writing $f_i = \{x_i, y_i\} = \{t, x_k\}$ ($t \in \{u_k, v_k\}$), either $x_i = t$ ($y_k t \in B(T_i) \cap B(Q_k^*) = \emptyset$) or $y_i = t$ ($y_k t \in B(Q_i^*) \cap B(Q_k^*) = \emptyset$), contradiction.

4. *Step 4 ($f_i \notin B(T_k)$ for $k \in \mathcal{B}_{\text{out}}$; `step4_bout_obstruction`):* Without loss of generality let $f_i = u_k x_k$. Since $f_k = x_k y_k \notin N$ is red, $C_k^{(v)} := \{f_k, v_k x_k, v_k y_k\} \in \mathcal{F}_2$ is disjoint from $U_i^{(u)}$ and $P' \setminus \{T_i, T_k\}$, forcing $B(Z_i) \cap B(C_k^{(v)}) \neq \emptyset$, i.e., $v_k y_k = \alpha z_i$ ($\alpha \in \{u_i, v_i\}$). Across all four orientations of $\{u_k, x_k\} = \{x_i, y_i\}$ and $\{v_k, y_k\} = \{\alpha, z_i\}$, one of $v_k x_k \in B(T_k) \cap B(Q_i^*)$, $e_k \in N \cap B(Q_i^*)$, $u_k y_k \in B(Q_k^*) \cap B(T_i)$, or $u_k y_k \in B(Q_k^*) \cap B(Q_i^*)$ occurs, each impossible.

Hence C_{core} covers $\mathcal{T}(G)$, proving (5″) (`Construction.opposite_elimination_cover`). □

Proof of Theorem 3. Combine (3), (4″), (5), (5″), (6a), and (6b) (`Construction.six_covers_81_29`) with integer weights $(10, 1, 8, 1, 2, 4)$ on cleared (4″)—i.e., $(10, 4, 8, 1, 2, 4)$ on the normalized bounds (`six_bounds_arithmetic_29_81`)—using $\hat{h} + h_0 = h$ and $t := \tau(G)$:

$$10\,(t + a) \le 30\,n, \qquad 8\,(t + c + d) \le 24\,n,$$
$$1\,(4t + 4c) \le 6\,n + 10\,a + 6\,b + \hat{h}, \qquad 1\,(t + 2c) \le 3\,n + \hat{h} + 2\,h_0,$$
$$2\,(t + b + 2d + \hat{h} + h_0) \le 6\,n + 6\,c + 2\,h_1, \qquad 4\,(t + b + h_1) \le 12\,n + 8\,c + 12\,d.$$

Summing gives $(10 + 4 + 8 + 1 + 2 + 4)t = 29\,t$ on the left, $(30 + 6 + 24 + 3 + 6 + 12)n = 81\,n$ on the right, exact cancellation on a $(10 - 10)$, b $(6 - 2 - 4)$, c $(4 + 8 + 2 - 6 - 8)$, d $(8 + 4 - 12)$, $\hat{h}$ $(2 - 1 - 1)$, and h_0 $(2 - 2)$, and left-hand slack $(4 - 2)h_1 = 2h_1 \ge 0$, proving (1.5) (`Construction.bound_81_29_with_slack`) and (1.4) (`tuza_bound_81_29_nat`, `tuza_bound_81_29`). Since $250 \times 81 = 20250 \le 20271 = 29 \times 699$, Theorem 3 also implies Theorem 1 directly. □

Remark 3.6 (Post-(5″) LP invariance at 81/29). *(Verified in `recompute.py`; not formalized in `lean/TuzaBound`.)* Each $T_{\text{old}} \in P_0 := \{T \in P : T \cap E(A) = \emptyset\}$ ($|P_0| = n - a$) cannot meet both $B(T_k)$ and $B(Q_k^*)$ for $k \in \mathcal{B}_{\text{out}}$ (else its third edge $f_k \in O'$ would be blue), and hence meets $B(P' \setminus \{T_k : k \in \mathcal{B}_{\text{out}}\})$ (else swapping hit $T_k \mapsto Q_k^*$ and adding T_{old} gives $(b, m + 1)$), proving $n - a \le 3m - b - 2\hat{h}$. Adjoining (4″), (5″), and $n - a \le 3m - b - 2\hat{h}$ to the depth-K Lemma 2.3 LP yields $\text{OPT}_K = \frac{81}{29}$ for every $K \in \{1, \ldots, 12\}$, attained at $(n, a, m, b, c, d, \hat{h}, h_0, h_1, \tau) = (29, 6, 23, 18, 6, 0, 6, 0, 0, 81)$.

4 Computational verification, LP duality, and sensitivity picture

In under 2 seconds of standard-library `int` and `fractions.Fraction` arithmetic (zero floating-point operations), `papers/tuza-triangle-cover/recompute.py` reproduces all LP primal–dual certificates, arborescence, chain-exchange, and Bell-partition enumerations, $H_{14,3,1}$ invariants, and small-graph censuses below. Outside the K_4 witnesses, Theorems 1 and 3, and Theorem 2(a), the diagnostics in this section are verified by `recompute.py`, not formalized in `lean/TuzaBound`.

4.1 2×2 LP synergy and primal–dual certificates

Neither Lemma 1.1 alone nor Lemma 1.2 alone improves upon Wang's $\frac{165}{59}$:

- **Ablation A ((4′) + Wang's merged (6)):** At $(a, m, b, c, d, h) = (\frac{12}{59}, \frac{47}{59}, \frac{39}{59}, \frac{12}{59}, 0, 0)n$, we have $h = 0$, so (3), (4′), (5), and merged (6) all equal $\frac{165}{59}n$, leaving the LP optimum at $\frac{165}{59}$.
- **Ablation B (Wang's (4) + depth-K unmerged Lemma 2.3):** By Proposition 3.5, $(a, m, b, c, d, h, h_1) = (\frac{12}{59}, \frac{47}{59}, \frac{36}{59}, \frac{12}{59}, 0, \frac{12}{59}, 0)n$ makes (3), Wang's (4), (5), (6a), (6c), and all deeper Lemma 2.3 bounds equal $\frac{165}{59}n$, again leaving the LP optimum at $\frac{165}{59}$.
- **Synergy in Theorem 1:** Inequality (6c) eliminates the $h = 0$ ray ($\frac{162}{59}n < \frac{165}{59}n$) while (4′) eliminates the $h = \frac{12}{59}n$ ray ($\frac{327}{118}n < \frac{165}{59}n$), shifting the optimal LP basis to $d = \frac{9}{250}n > 0$ and OPT $= \frac{699}{250} = 2.796$ (with $K = 2$ dual multipliers $\frac{393}{250}$ on $a + m \le 1$ and $(85, 34, 80, 12, 39)/250$ on (3)–(6c); Table 1 and Figure 3).
- **Opposite-edge elimination ladder to Theorem 3:** Inequality (5″) cuts off the $h = 0$ ray of Theorem 1 (forcing the $h = 0$ LP slice down to $\frac{36}{13} \approx 2.769231$) and lowers the global LP optimum through progressive relaxations $\frac{3687}{1319}$ ($+3h$ unsplit), $\frac{109}{39}$ ($+2h$ unsplit), and $\frac{4071}{1457}$ (Depth-1 without all-depth arborescences) down to OPT $= \frac{81}{29} \approx 2.793103$ using only elementary (6a) and (6b) (Table 1 and Figure 3).

Table 1: Exact rational primal–dual LP progression, 2×2 ablation, and opposite-edge elimination ladder (certified with zero duality gap over `fractions.Fraction` in `recompute.py`).

Transversal / LP system	Exact c_*	Decimal	$c_* - 2$	Verification scope
Trivial packing support $E(P)$	3	3.000000	1.000000	Lean 4 (`graph_trivial_bound`)
Haxell (1999) 4-bound LP [6]	66/23	2.869565	0.869565	`recompute.py` $(5, 2, 12, 4)$
Yi (2026) rational 2-colorable [21]	63/22	2.863636	0.863636	`recompute.py` $(5, 2, 11, 4)$
Haxell 4-lemma limit ($\tau(\mathcal{S}) \le 2\nu(\mathcal{S})$)	54/19	2.842105	0.842105	`recompute.py` $(5, 2, 8, 4)$
Wang (2026) (3) + (4) + (5) + (6) [18]	165/59	2.796610	0.796610	Lean 4 (`four_covers`)
Ablation A: (3) + (4′) + (5) + (6_{merged})	165/59	2.796610	0.796610	`recompute.py` ($h = 0$ ray)
Ablation B: (3) + (4) + depth-K Lem. 2.3	165/59	2.796610	0.796610	`recompute.py` ($h = \frac{12}{59}n$ ray)
Depth-1 pigeonhole + (6a) + (6c)	4611/1649	2.796240	0.796240	`recompute.py` (Rem. 3.3)
(3) + (4′) + (5) + (6_{merged}) + (6b)	288/103	2.796117	0.796117	`recompute.py` $(35, 14, 33, 20, 1)$
Depth-1 unit-chord secants $[2, 3], [3, 4]$	652959/233525	2.796099	0.796099	`recompute.py` (Rem. 3.3)
Theorem 1: (3) + (4′) + (5) + (6a) + (6c)	**699/250**	**2.796000**	**0.796000**	**Lean 4 (`tuza_bound_nat`)**
Theorem 2(a): pre-(5″) depth-K LP	**699/250**	**2.796000**	**0.796000**	**Lean 4 (`FixedPoint.lean`)**
Level 1: (3) + (4′) + (5) + ($5'_3$) + (6a) + (6c)	3687/1319	2.795299	0.795299	`recompute.py` ($+3h$ unsplit)
Level 2: (3) + (4′) + (5) + ($5'_2$) + (6a) + (6c)	109/39	2.794872	0.794872	`recompute.py` ($+2h$ unsplit)
Level 3: Depth-1 + (5″) (0 arborescences)	4071/1457	2.794097	0.794097	`recompute.py`
Theorem 3: (3) + (4″) + (5) + (5″) + (6a) + (6b)	**81/29**	**2.793103**	**0.793103**	**Lean 4 (`tuza_bound_81_29_nat`)**
Post-(5″) depth-K LP ($1 \le K \le 12$)	81/29	2.793103	0.793103	`recompute.py` (Rem. 3.6)
$h = 0$ slice with (4″) + (5″)	36/13	2.769231	0.769231	`recompute.py`

4.2 Finite arborescence, chain-exchange, and Bell-partition stress tests

`recompute.py` performs four finite stress checks on Lemmas 3.1, 3.2, and 1.3:

1. *Unlabeled rooted trees ($1 \le s \le 7$):* Across all $1, 1, 2, 4, 9, 20, 48$ (85 total) canonical rooted arborescences on $s = 1, \ldots, 7$ vertices, $\min_{\mathcal{T}} \sum_{v \in \mathcal{T}} \frac{1}{8} 4^{-\deg^-(v)} = \frac{3}{32} + \frac{s}{32}$

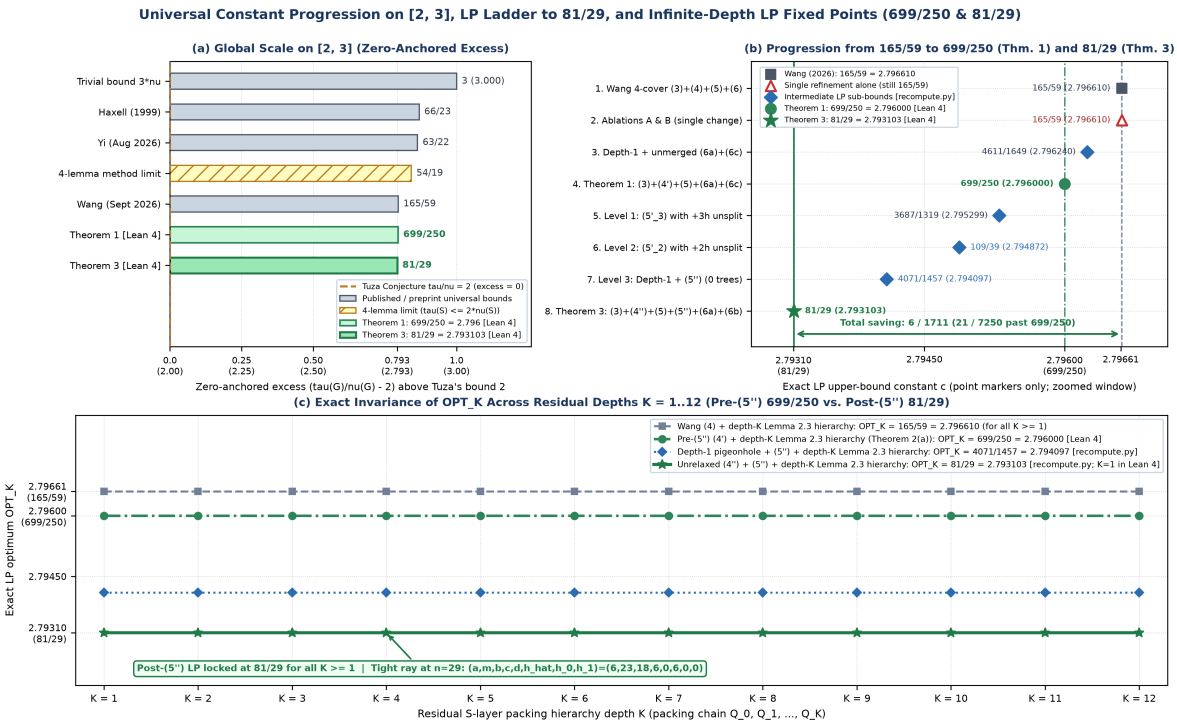


Figure 3: Quantitative comparison of universal bounds and LP relaxations: (a) zero-anchored excess $\tau/\nu - 2 \in [0, 1]$ on $[2, 3]$; (b) point-marker progression from $\frac{165}{59}$ through $\frac{699}{250}$ (Theorem 1) to $\frac{81}{29}$ (Theorem 3); (c) exact invariance of OPT_K across residual depths $K = 1, \ldots, 12$ for both the pre-$(5'')$ LP ($\frac{699}{250}$) and post-$(5'')$ LP ($\frac{81}{29}$).

$(1/8, 5/32, 3/16, 7/32, 1/4, 9/32, 5/16$, achieved on directed paths, up to $\frac{24577}{32768}$ on the 6-leaf star at $s = 7$), with single-vertex secant slacks $0, 0, \frac{9}{128}, \frac{81}{512}, \frac{513}{2048}, \frac{2817}{8192}$ at $k = 0, \ldots, 5$.

2. *All vertex-identification patterns ($r \leq 4$):* Across all set partitions of the $2r + 3$ slots $(x_1, y_1, u_1, v_1, \ldots, u_r, v_r, z)$, all $1, 3, 17, 71$ (92 total) valid canonical patterns for $r = 1, 2, 3, 4$ (and all $1, 6, 68$ oriented swap patterns for $r = 1, 2, 3$) admit a disjoint toggle $(\varepsilon_1, \ldots, \varepsilon_r) \in \{0, 1\}^r$ (0 obstructions).
3. *Exhaustive Bell-partition check of Lemma 1.3:* Enumerating all $B_{10} = 115975$ set partitions of the 10 slots $(u_i, v_i, x_i, y_i, z_i, u_k, v_k, x_k, y_k, z_k)$ in Step 3 (28 valid configurations) and all $B_9 = 21147$ set partitions of the 9 slots in Step 4 without assuming $u_k \leftrightarrow v_k$ symmetry (12 valid configurations) finds 0 surviving configurations.
4. *Exact $2^{|V|}$ Boolean cuts (Table 2):* Exact cut enumeration on four helper trees and a 9-vertex K_6 directed 3-cycle plus coupled sink book confirms universal triangle coverage and exact saving $\geq \frac{3}{32} + \frac{s}{32}$.

4.3 Small-ν census and extremal gluing dichotomy

Across all $2, 6, 21, 112$ connected non-isomorphic simple graphs on $|V| = 3, 4, 5, 6$ vertices, exactly $0, 1, 2, 5$ satisfy $\tau(G) = 2\nu(G)$ ($\max \tau/\nu = 2$). Table 3 contrasts 13 benchmark graphs and gluings across $\nu \in \{1, 2, 3, 4, 7\}$ (for $\nu \leq 4$, $\tau \leq 2\nu$ holds generally [13]): cut-

Table 2: Exact $2^{|V|}$ Boolean random-cut expectations on explicit helper configurations (`recompute.py`; every cut covers all type-2 and type-3 triangles).

Configuration	$\lvert V\rvert$	b	$\hat{h}$	$\mathbb{E}[\lvert D_{\mathrm{red}}\rvert]$	Lem. 1.1 $\frac{b}{2}+\frac{\hat{h}}{4}$	Wang $\frac{b}{2}+\frac{3\hat{h}}{8}$	Exact saving
Isolated K_4 book ($s=1$)	4	1	1	3/4	3/4 (tight)	7/8	1/8
Helper chain $j_1 \to i_1$ ($s=2$)	6	2	1	39/32	5/4	11/8	5/32
Helper chain $j_2 \to j_1 \to i_1$ ($s=3$)	8	3	1	27/16	7/4	15/8	3/16
Two-leaf star $j_1, j_2 \to i_1$ ($s=3$)	8	3	1	207/128	7/4	15/8	33/128
K_6 directed 3-cycle + coupled sink	9	4	1	69/32	9/4	19/8	7/32

vertex gluings and the 2-connected C_4-necklace of four K_4's ($|V|=12, \nu=4, \tau=8$) preserve $\tau=2\nu$, whereas edge and triangle gluings and the 2-connected C_3-ring of three K_4's ($|V|=9, |\mathcal{T}|=13, \nu=4, \tau=6$, via the central 13th triangle) drop strictly below 2ν, and K_7 ($\nu=\tau^*=7, \tau=9$) exhibits fractional-to-integral ratio $\tau/\tau^*=\frac{9}{7}$.

Table 3: Exact invariants $(\nu(G), \tau^*(G), \tau(G))$ on 13 benchmark graphs and $H_{14,3,1}$ (`recompute.py`).

Graph G	$\lvert V\rvert$	$\lvert E\rvert$	$\lvert\mathcal{T}(G)\rvert$	$\nu(G)$	$\tau^*(G)$	$\tau(G)$	$\tau(G)/\nu(G)$
K_4 (`nu_complete4`, `tau_complete4` in Lean 4)	4	6	4	1	2	2	2
K_5	5	10	10	2	10/3	4	2
K_7 (Fano STS(7) packing)	7	21	35	7	7	9	9/7
Octahedron $K_{2,2,2} = K_6 - M_3$	6	12	8	4	4	4	1
Wheel W_5	6	10	5	2	5/2	3	3/2
$K_4 \vee_v K_4$ (cut-vertex bowtie)	7	12	8	2	4	4	2
$K_4 \cup_e K_4$ (shared-edge diamond)	6	11	8	2	3	3	3/2
$K_5 \vee_v K_4$	8	16	14	3	16/3	6	2
$K_5 \cup_e K_4$	7	15	14	3	5	5	5/3
$K_5 \cup_\triangle K_4$	6	13	13	3	4	4	4/3
$K_5 \vee_v K_5$	9	20	20	4	20/3	8	2
C_3-ring of three K_4's (2-connected)	9	18	13	4	6	6	3/2
C_4-necklace of four K_4's (2-connected)	12	24	16	4	8	8	2
Colored $\mathcal{S}$-gadget $H_{14,3,1}$ (Thm. 2(b))	32	62	32	14	14	14	1

5 Formal verification in Lean 4

Library overview, toolchain, and credit to Sichen Wang. The Lean 4 library `TuzaBound` (`lean/TuzaBound.lean` and 27 modules under `lean/TuzaBound/`, 28 `.lean` files total) compiles under Lean 4 `v4.35.0` and Mathlib `v4.35.0` with `autoImplicit = false` and `relaxedAutoImplicit = false` (audited at commit `9d33b74fd28f4ab6 2793fcd199f65c78a6ae8e87` and baseline `4f10f6c7e859b4beb746378a1b756990532368fe` in `lean/TuzaBound/audit.md`). It ports and extends Sichen Wang's MIT-licensed Lean 4 (`v4.32.0`) formalization `sichen-wang/tuza-lean` [18, 20] into `v4.35.0` `module` mode under `namespace TuzaBound`, preserving `Copyright (c) 2026 Sichen Wang` in all 28 file headers and Wang's verbatim MIT license in `lean/TuzaBound/LICENSE`.

Kernel verification metrics and specification audit. Purging all 448 cached `TuzaBound` build files and rebuilding from source compiles all 28 modules (3188 Lake jobs) with 0 warnings and 0 errors:

- Across the 27 non-`Verify` library files (out of 28 total `.lean` files including `Verify.lean`) there are 544 **public declarations** (459 public `theorem`s and 85 definitions, structures, and inductives) and 0 **`private` declarations**. In `lean/TuzaBound/Verify.lean`, 122 `#check` commands and 113 **receipted `#print axioms` declarations** (109 public theorems plus the 4 definitions `nu`, `tau`, `graphEdges`, and `graphTriangles`) certify `lean/TuzaBound/receipt.json` with `allow_native_decide = false` and `axioms_stdout_sha256` equal to `0dda8074ce94f48477f8f98dc7d37bc6ba7bcbf1d2ec4103c8f02d18f7e2c3a3`.
- All 113 receipted declarations (8 using `[propext, Quot.sound]` and 105 using `[propext, Classical.choice, Quot.sound]`) and all 431 remaining public declarations (350 helper theorems and 81 auxiliary definitions) depend solely on Lean 4's three standard axioms, with **zero** `sorry`, `admit`, custom `axiom`, `unsafe`, `extern`, or `native_decide`. All $\mathbb{N}$-inequalities in (3)–(6c) are formulated purely with addition and multiplication (zero `Nat.sub` truncation), with exact additive partition $\hat{h} + h_0 = h$ (`helperRoots_add_privZeroPages_card`).
- Line-by-line diffing against all 22 upstream modules of `sichen-wang/tuza-lean` [20] (`lean/TuzaBound/audit.md`) confirms that 11 foundational modules (including `Construction/OppositeEdges.lean`, `Basic/Graph.lean`, `Basic/Packing.lean`, and `Basic/Geometry.lean`) are character-for-character identical in body after module headers, leaving `IsTriangle`, `IsPacking`, `IsCover`, `nu`, and `tau` 100% unmodified; 3 modules drop only `private` under `module` visibility; 7 modules are purely additive (replacing `decide +kernel` in Boolean averages with algebraic proofs and adding (6a)–(6c) and `six_bounds_arithmetic_29_81`); and 7 top-level or new files (`Main.lean`, `Probability/Arborescence.lean`, `Construction/HelperArborescence.lean`, `Construction/OppositeElimination.lean` [1549 lines], `FixedPoint.lean`, `Verify.lean`, and `TuzaBound.lean`) prove Lemmas 1.1–1.3, Theorems 1, 2(a), and 3, Corollary 3.4, and K_4 non-vacuity without requiring `[Nonempty V]` or `[DecidableRel G.Adj]`. In `FixedPoint.lean`, `IsValidInfiniteHierarchyPoint` omits the redundant disjoint blue-edge budget ($597 \leq 750$ on the primal witness), strengthening `infinite_depth_lp_upper` onto a larger polyhedron.

Summary of what IS versus IS NOT formalized in `lean/TuzaBound`. Table 4 records the complete formalized scope: Theorems 1 and 3, Lemmas 1.1–1.3 and 3.1–3.2, Corollary 3.4, K_4 non-vacuity, and Theorem 2(a) are pure-kernel Lean 4 theorems. By contrast, Theorem 2(b) ($H_{14,3,1}$ and the $n = 250$ obstructions $\frac{1281}{2} < 699$ and $|C^{(5)}| \leq 669 < 699$), Proposition 3.5, Remarks 3.3 and 3.6, and the enumerations of Section 4 are proved in Section 3 and verified in exact rational arithmetic by `recompute.py`, not formalized in `lean/TuzaBound`.

Table 4: Mapping between paper statements and Lean 4 declarations in `lean/TuzaBound`. Unparenthesized identifiers are public `theorem` declarations audited in `Verify.lean`; `(def)` and `(structure)` mark foundational definitions and structures.

Paper statement	Module	Lean 4 declaration(s)	Scope
Def. 2.1 ($\mathcal{T}(G), \nu, \tau$) & K_4 ($\nu = 1, \tau = 2$)	`Basic/Graph.lean`, `Basic/Packing.lean`, `Main.lean`	`IsPacking` (def), `IsCover` (def), `graphEdges` (def), `graphTriangles` (def), `nu` (def), `tau` (def), `triangleEdges_mem_graphTriangles_iff`, `graph_cover_iff_adj`, `exists_graph_minimumCover`, `nu_complete4`, `tau_complete4`	Pure kernel
Lems. 2.1–2.3	`Books/Cover.lean`, `Colored/Exchange.lean`, `Colored/Covers.lean`	`unique_residual_triangle`, `exists_cover_of_small_common_spine_books`, `first_colored_cover`, `second_colored_cover`, `colored_bound`, `recursive_colored_cover`	Pure kernel
Haxell–Wang tuple, eqs. (3) & (5)	`Construction/Setup.lean`, `BookCover.lean`, `Main.lean`	`Construction` (structure), `RefinedConstruction` (structure), `exists_construction`, `exists_refinedConstruction`, `Construction.first_cover`, `Construction.full_book_cover`	Pure kernel
Lem. 3.1 (basins, degree sum, secant (3.3))	`Probability/Arborescence.lean`	`Arborescence.reachIn` (def), `Arborescence.basin` (def), `Arborescence.childrenOf` (def), `Arborescence.reachIn_sink_unique`, `Arborescence.exists_chain_from_sink`, `Arborescence.basin_disjoint`, `Arborescence.sum_childrenOf_card`, `Arborescence.fourth_pow_secant`, `Arborescence.global_prob_sum_le`	Pure kernel
Lemma 1.1 ((4″) & (4′))	`Probability/Average.lean`, `Cut.lean`, `Construction/HelperArborescence.lean`	`cutCover` (def), `HelperArborescence.helperNext` (def), `HelperArborescence.helperRoots` (def), `HelperArborescence.basin_third_vertex_child`, `Construction.helper_arborescence_random_cover_unrelaxed`, `Construction.helper_arborescence_random_cover`	Pure kernel
Lemma 1.2 ((6a)–(6c))	`Construction/ColoredCover.lean`	`Construction.unmerged_cover_6a`, `Construction.unmerged_cover_6b`, `Construction.unmerged_cover_6c`, `Construction.elementary_colored_covers`, `Construction.unmerged_colored_covers`	Pure kernel
Theorem 1 & Cor. 3.4 ($250\,\tau \le 699\,\nu$)	`Basic/Arithmetic.lean`, `Main.lean`	`five_bounds_arithmetic`, `Construction.five_covers`, `Construction.bound_with_private_slack`, `tuza_bound_nat`, `tuza_bound`, `exists_small_triangle_transversal`, `wang_bound_from_699_250`	Pure kernel ($\forall G$)
Theorem 2(a) (pre-(5″) LP $= 699/250$)	`FixedPoint.lean`	`IsValidInfiniteHierarchyPoint` (structure), `book2_le_book3`, `fixed_point_recurrence`, `infinite_depth_lp_upper`, `infinite_depth_lp_attained`, `infinite_depth_lp_exact_699_250`	Pure kernel ($\mathbb{Q}$)
Lemma 1.3 & Theorem 3 ($29\,\tau \le 81\,\nu$)	`Construction/OppositeElimination.lean`, `Basic/Arithmetic.lean`, `Main.lean`	`OppositeElimination.privZeroPages` (def), `IsLexMaximumPacking.no_one_to_two`, `IsLexMaximumPacking.no_two_to_three`, `OppositeElimination.step3_nonprivate_obstruction`, `OppositeElimination.step4_bout_obstruction`, `Construction.opposite_elimination_cover`, `six_bounds_arithmetic_29_81`, `Construction.bound_81_29_with_slack`, `tuza_bound_81_29_nat`, `tuza_bound_81_29`, `exists_small_triangle_transversal_81_29`	Pure kernel ($\forall G$)

6 Prior work and open questions

6.1 Literature context and read-depth disclosure

Every reference cited here is logged in `references-checked.md` with its verified identifier, access date, and read depth. Beyond Krivelevich's fractional bounds $\tau(G) \le 2\nu^*(G)$ and $\nu^*(G) \le 2\nu(G)$ [11] [to check] (extended to weighted packings in [2] [to check] and fractional r-uniform hypergraphs in [19] [to check]; compare [1] [to check]), universal constants $\tau(G) \le c\,\nu(G)$ for *all* finite simple graphs advanced from 3 to $\frac{66}{23}$ in Haxell (1999) [6] [to check], $\frac{63}{22}$ in Yi (August 2026) [21], and $\frac{165}{59}$ in Sichen Wang (September 2026) [18, 20]. Restricted graph classes where $\tau(G) \le 2\nu(G)$ (or better) is proved include tripartite graphs ($\le 1.956\,\nu(G)$, [7] [to check]), K_4-free planar graphs ($\le \frac{3}{2}\nu(G)$, [8] [to check]), random graphs [9] [to check], $\nu(G) \le 4$ [13], $\Delta(G) \le 7$ [5] [to check], $\Delta(G) \le 10$ [10] [to check], high edge density [12] [to check] (compare OEIS A002620 [15] and A001839 [14] for K_n), bounded overlap cores and average degree [16] [to check], and K_4-free graphs of book size at most 3 ($\frac{353}{177}\nu(G)$, [22] [to check]). Searches across arXiv, zbMATH Open, Zenodo, and GitHub returned zero prior appearances of $\frac{81}{29}$, $\frac{699}{250}$, $\frac{4071}{1457}$, $\frac{109}{39}$, $\frac{3687}{1319}$, $\frac{4611}{1649}$, or $\frac{652959}{233525}$ for Tuza's conjecture.

Read-depth disclosure. We read in full the 5-page PDF of Wang's `arXiv:2609.13831v1` [18], all 22 Lean 4 modules and the MIT license of `sichen-wang/tuza-lean` [20], the 4-page PDF of Yi's `arXiv:2608.23010v1` [21], Erdős Problem #167 [3], `TuzaConjecture.lean` in [4], OEIS entries A001839 [14] and A002620 [15], and repository READMEs and statements of [13, 22]. For Krivelevich (1995) [11] and Haxell (1999) [6], we verified metadata, DOI/zbMATH records, and exact theorem statements via [18, 20, 21], while the printed *Discrete Mathematics* articles remain unread [to check]. For preprints [5], [10], [12], [16], and [19], arXiv metadata and abstracts were read [full PDFs to check]; for papers [1], [2], [7], [8], and [9], indexed zbMATH search records were read [full bibliographic details and texts to check]; and Tuza (1981, 1990) [17] was cited via [18, 21] [primary sources to check]. Subscription databases (MathSciNet, Scopus, Google Scholar) also remain [to check].

6.2 Open questions

1. **Simple-graph realizability of the $\frac{81}{29}$ six-cover limit (conjectured):** Writing $m_6(G)$ for the minimum size among the six transversals of (3), (4″), (5), (5″), (6a), and (6b), do finite simple graphs free of accidental triangles realize $m_6(G)/\nu(G) = \frac{81}{29} - o(1)$ along the ray $(n, a, m, b, c, d, \hat{h}, h_0, h_1) = (29, 6, 23, 18, 6, 0, 6, 0, 0)$, or is $\sup_{\nu(G)\ge 1} m_6(G)/\nu(G) < \frac{81}{29}$?
2. **Breaking below $\frac{81}{29}$ via global old-triangle structure:** At $(n, a, m, b, c, \hat{h}) = (29, 6, 23, 18, 6, 6)$, can structural constraints on how the $n - a$ untouched old triangles P_0 intersect W and $B(P' \setminus B^*)$ yield a universal transversal bound strictly below $\frac{81}{29}\nu(G)$?
3. **Multi-stage random cuts beyond (P', Q):** Can dependent or three-page random cuts adapted to (P, A, P', Q, Q_1) reduce the $\frac{3}{2}b$ term in (4″)?

7 Data availability, reproducibility, and use of AI

Repository paths and rebuild commands. All formal proofs, exact-arithmetic verifiers, figure scripts, and ledgers live under `lean/TuzaBound/` and `papers/tuza-triangle-cover/` (`recompute.py`, `recompute.out`, `figures/make_*.py`, `claims.md`, `references-checked.md`, `receipt.json`) and rebuild from the repository root via:

```
(cd lean && lake build TuzaBound && lake env lean TuzaBound/Verify.lean)
python3 -B papers/tuza-triangle-cover/recompute.py \
  -o papers/tuza-triangle-cover/recompute.out
(cd papers/tuza-triangle-cover/figures && \
  python3 -B make_discovery_architecture.py && \
  python3 -B make_helper_arborescence_secant.py && \
  python3 -B make_lp_ablation_and_fixed_point.py)
python3 tools/check_paper_top20.py -p papers/tuza-triangle-cover -b \
  -o papers/tuza-triangle-cover/receipt.json
```

Licensing credit and use of artificial intelligence. Foundational Lean 4 modules in `lean/TuzaBound/` are ported and extended from Sichen Wang's MIT-licensed repository `sichen-wang/tuza-lean` [20] (`Copyright (c) 2026 Sichen Wang`; full license text in `lean/TuzaBound/LICENSE`). The mathematical refinements proving $\tau(G) \le \frac{699}{250}\nu(G)$ and $\tau(G) \le \frac{81}{29}\nu(G)$, the infinite-depth LP and $H_{14,3,1}$ barrier analysis, the Lean 4 extensions in `lean/TuzaBound`, `recompute.py`, the figures, and this LaTeX scaffold were generated and audited by autonomous AI research agents under fail-closed verification gates. No human mathematician has yet reviewed this draft; the authors assume full responsibility for checking every proof step, completing all [to check] lookups, and rewriting the manuscript prior to submission.

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