Real point sets with few lines through every point

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Abstract

How many distinct connecting lines must pass through the busiest point of a noncollinear set in the real plane? We construct, for every $n \geq 8$, a set of n points in which each point lies on at most $\lfloor n/2 \rfloor$ connecting lines. A punctured triangular grid supplies the residue class left open by Akiyama, Ito, Kobayashi and Nakamura. A second construction uses eight directions of parallel lines in the projective dual. It gives three infinite families of odd sizes whose busiest points lie on exactly $(n-5)/2$ lines. Removing a specified diagonal preserves that maximum at the adjacent even size. An exact interval-union count supplies three further pairs of maxima. The examples contradict Grünbaum's eventual formula $2\lfloor n/4 \rfloor$ in twelve residue classes modulo sixty. We turn the infinite upper bounds into finite calculations by grouping intersection heights by their residues and bounding each group by an interval. Lean 4 checks the all-size bound, the first three odd-family maxima and the residue-class upper bounds, including the finite certificates. The remaining equalities follow from exact interval unions and shared intersections. Whether a universal constant bounds the deficit below half for real configurations remains open.

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1 Introduction and main results

Mark some points on a sheet of paper and draw the line through every pair. Three points on one line save two strokes. Can enough such coincidences coexist that even the busiest point sees fewer than half as many lines as there are points? Write $t(P)$ for the largest number of distinct spanned lines through one point of P, and $t(n)$ for its minimum over noncollinear n-point sets $P \subset \mathbb{R}^2$.

Theorem 1. *For every $n \geq 8$, $t(n) \leq \lfloor n/2 \rfloor$. For $n = 12j + 11$ with $j \geq 2$, an explicit rational configuration has $t(P) = 6j + 5$.*

In particular every odd $n \geq 9$ admits $t(P) < \lceil n/2 \rceil$. Akiyama, Ito, Kobayashi and Nakamura left the cases $12j + 11$ with $j \geq 4$ open [2]. The construction below covers that whole progression, and the proof also gives a realization for every other size.

Theorem 2. *Set $G(n) = 2\lfloor n/4 \rfloor$. For all sufficiently large n in the residue classes $0, 1, 8, 9, 52, 53 \pmod{60}$ there are rational configurations with $t(P) = G(n) - 2$. For classes $4, 5, 44, 45, 48, 49 \pmod{60}$ there are rational configurations with $t(P) = G(n) - 1$.*

Grünbaum's proposed eventual value was $G(n)$ [4]. In each case the construction below identifies a point attaining the stated maximum.

The six pairs of constructions in Theorem 2 begin at different sizes. Both configurations in each row attain the displayed maximum.

Construction	Range	Odd size n	Even size	$t(P)$ in the two constructions
A	$s \geq 0$	$60s + 53$	$n - 1$	$30s + 24 = G(n) - 2$
B	$s \geq 0$	$60s + 69$	$n - 1$	$30s + 32 = G(n) - 2$
$C = B \setminus D_8$	$s \geq 1$	$60s + 61$	$n - 1$	$30s + 28 = G(n) - 2$
$B \setminus D_4$	$s \geq 1$	$60s + 65$	$n - 1$	$30s + 31 = G(n) - 1$
$A \setminus D_8$	$s \geq 2$	$60s + 45$	$n - 1$	$30s + 21 = G(n) - 1$
$A \setminus D_4$	$s \geq 2$	$60s + 49$	$n - 1$	$30s + 23 = G(n) - 1$

For the first three odd rows $n/2 - t(P) = 5/2$. In the corresponding even rows the deficit is 2. The table gives upper bounds on the minimum $t(n)$ in all cases. The Strong Dirac conjecture asks whether $t(n) \geq n/2 - c$ for some absolute c [1, 5].

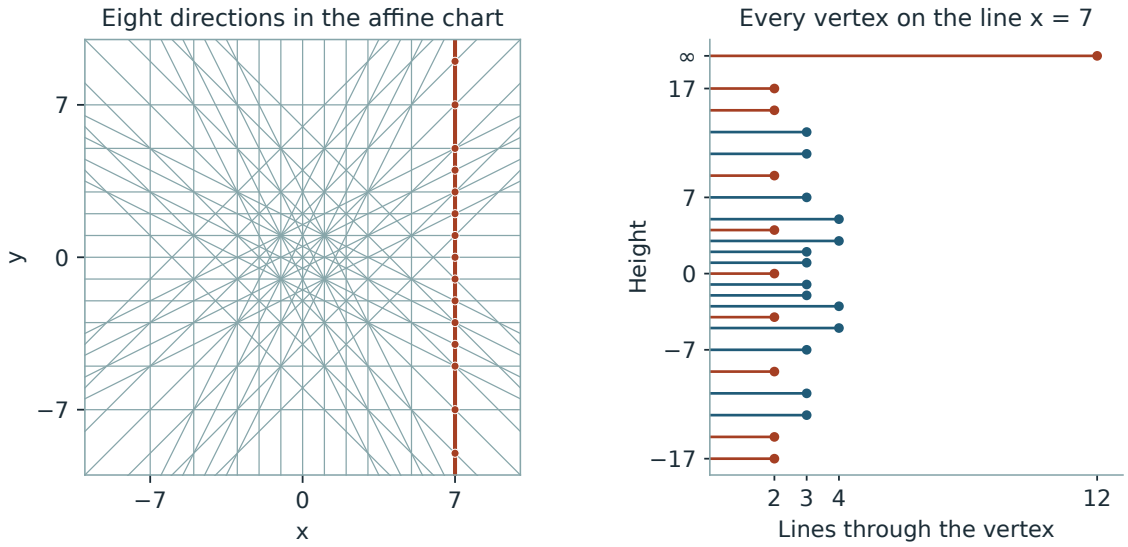


Figure 1. The smallest eight-direction witness. Each dot on the orange line at left is a distinct vertex in the displayed window. The plot at right includes all heights and the shared point at infinity. Searching for coincident crossings led to eight directions. Residue intervals and ordinary vertex deletions prove the bounds at every scale. A separate punctured grid proves the half bound. Both upper-bound arguments are checked in Lean.

Instant witness check. On $x = 7$ the finite height from $px + qy = c$ is $(c - 7p)/q$ when $q \neq 0$. This box prints the number of distinct vertices there, including the one at infinity.

```
from fractions import Fraction
def I(b): return range(-b, b+1, 2)
bands = [(0, 1, 7), (0, 1, 2), (1, 1, 6), (1, -1, 6),
         (1, 2, 3), (1, -2, 3), (2, 1, 3), (2, -1, 3)]
heights = {Fraction(c-7*p, q) for p, q, b in bands for c in I(b)}
print(len(heights) + 1)
```

2 Dual coordinates and the eight directions

A nonzero real triple specifies a projective point up to scaling. Duality sends each point to a line and the lines through it to the vertices on that dual line. Thus an arrangement $\mathcal{L}$ of n distinct projective lines, not all concurrent, gives a point set with $t(P) = \max_{L\in\mathcal{L}} r(L)$, where $r(L)$ counts distinct vertices on L. Parallel affine lines meet once at infinity. We usually include the line at infinity in $\mathcal{L}$.

The resulting point configuration can always be put in an affine plane. For each point $(x : y : z)$ choose the same integer T so that $w = z + Tx + T^2y$ is nonzero at every point. Only finitely many T are excluded. The invertible map $(x, y, z) \mapsto (x, y, w)$ followed by division by w preserves all incidences. Rational line coefficients therefore furnish rational Euclidean examples.

Let $m(v)$ be the number of arrangement lines through a vertex v. On a fixed L, every other line is counted once at its intersection with L, so

$$\sum_{v\in L}(m(v) - 1) = n - 1, \qquad \sum_{v\in L}(m(v) - 3) = n - 1 - 2r(L).$$

For $A(0)$ the 24 vertices on $x = 7$ have multiplicities $2, 3, 4, 12$ with frequencies $9, 10, 4, 1$. The nine ordinary vertices contribute -9 in the second sum. The four quadruple vertices and the point at infinity together contribute 13. The excess 4 gives $r(L) = (52 - 4)/2 = 24$.

The arrangements. Write $I(d) = \{-d, -d + 2, \ldots, d\}$. In addition to the line at infinity, take the lines $px + qy = c$ in the following pencils:

$$\begin{array}{rll} \text{axes} & (p, q) = (1, 0), (0, 1) & c \in I(a) \cup I(e), \\ \text{diagonals} & (p, q) = (1, 1), (1, -1) & c \in I(b), \\ \text{knights} & (p, q) = (1, \pm 2), (2, \pm 1) & c \in I(m). \end{array}$$

The normals of the last four pencils are chess-knight moves. For A use $(a, e, b, m) = (9s + 7, 3s + 2, 8s + 6, 5s + 3)$. For B add $(2, 0, 2, 2)$. The two axis intervals have opposite parities, giving $n = 2a + 2e + 2b + 4m + 11$. The deletion D_8 consists of the eight knight lines with $c = \pm(m - 2)$, and D_4 keeps just the four with normals $(1, \pm 2)$ in that deletion set. All these deletions leave both axis directions and the line at infinity.

Choosing a direction in parameter space. An explanation for the ratios in A and B comes from four linear design conditions. In the variable order (a, e, b, m) set

$$B_1 = 3a - 2e - 2b - m, \quad B_2 = -a + 3e, \quad B_3 = -3a + 4b - m, \quad B_4 = -15a - 10b + 43m.$$

Their weighted sum $27B_1 + 18B_2 + 16B_3 + B_4$ vanishes identically. Consequently $B_i \le 0$ for all four forces equality throughout. Solving $B_2 = B_3 = B_1 = 0$ gives $(a, e, b, m) = u(9, 3, 8, 5)$. This observation identifies a ray for designing the examples. Section 4.1 derives the conditions from leading boundary counts and explains their range of validity.

3 Proofs of the two constructions

3.1 Where the punctures go

Let $K = 2j+2 \geq 6$. Start with $x = i, y = i, x+y = i$ for integers $-K \leq i \leq K$. Remove $i = \pm(K-1)$ from all three pencils and $i = 1$ from the first two. Denote the remaining diagonal indices by J_Z and the remaining axis indices by J. Add the three lines $x - y = 0,\ x + 2y = 0,\ 2x + y = 0$ and the line at infinity. The result S_K has $2|J| + |J_Z| + 4 = 6K - 1$ lines.

Lemma 3. *Every line of S_K contains at most $3K - 1$ vertices. On $x = K$ this bound is attained.*

Proof. On $x = i$ the finite heights are exactly $J \cup (J_Z - i) \cup \{i, -2i, -i/2\}$. On $x + y = i$ the finite abscissae are $J \cup (i - J) \cup \{i/2, 2i, -i\}$. In either case the integers lie in an interval of $2K + 1 + |i|$ consecutive integers. There is at most one extra half-integer, present only for odd i, and a single point at infinity. This proves the bound for $|i| \leq K - 4$.

At $|i| = K - 3, K - 2$, the axis interval misses $\operatorname{sgn}(i)(K - 1)$, and the diagonal interval misses $-\operatorname{sgn}(i)(K - 1)$. Substituting these offsets in the preceding sets verifies the holes. For $K - 3$ the possible half-integer costs one vertex. For $K - 2$ all heights are integral. Each bound is $3K - 1$. At $i = \pm K$ the interval has $3K + 1$ integers. Three holes suffice:

	$i = K$	$i = -K$
$x = i$	$1,\ K-1,\ 1-2K$	$1,\ 1-K,\ 2K-1$
$x + y = i$	$1,\ 1-K,\ 2K-1$	$1,\ K-1,\ 1-2K$

The horizontal case follows by exchanging x, y. The line at infinity has six directions. On each of the three added lines one may scale an affine coordinate so all finite values lie in $2J_Z \cup J_Z$. For instance use $2y$ on $x = y$ and $-2y$ on $x + 2y = 0$. The even elements of J_Z already belong to $2J_Z$, leaving at most $K - 2$ odd elements. Including infinity gives at most $(2K - 1) + (K - 2) + 1 = 3K - 2$.

On $x = K$ the horizontals give all J. Diagonals add $[-2K, -K - 1] \setminus \{1 - 2K\}$ and the further height $1-K$. These three sets are disjoint. With infinity they give $(2K-2)+(K-1)+1+1 = 3K-1$ distinct vertices. □

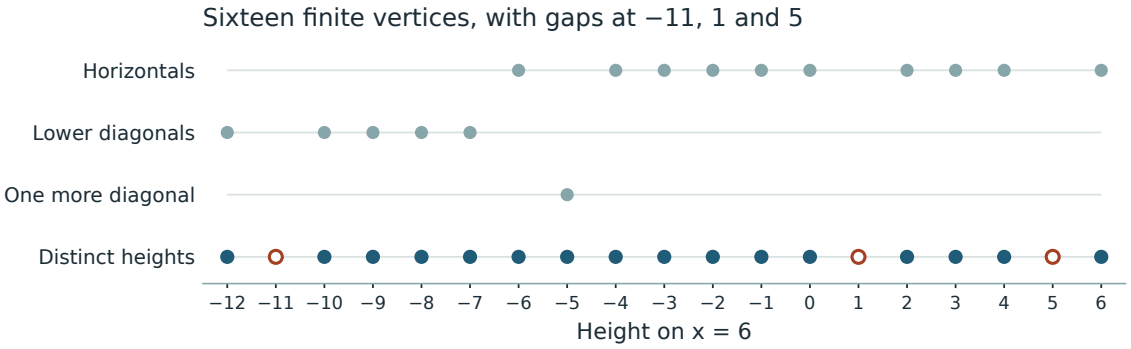


Figure 2. A complete transversal for the first punctured example, $K = 6$. Each mark in an upper row supplies one of the occupied heights below. The unfilled circles are the three missing integers. The shared point at infinity gives the seventeenth vertex on this line of a 35-line arrangement.

3.2 Covering every size

For a surviving line L, deleting other lines can only remove vertices. In particular each deleted line meeting L in an ordinary vertex removes one distinct vertex there. We use the following constructions in pairs, deleting the line at infinity for the smaller size.

Construction	Sizes	Range	Common upper bound
Square grid and two diagonals	$4u+4, 4u+5$	$u \geq 1$	$2u+2$
Full triangular grid and three medians	$12k+6, 12k+7$	$k \geq 1$	$6k+2$
Full grid minus four lines	$12k+2, 12k+3$	$k \geq 3$	$6k+1$
Punctured grid S_K	$12j+10, 12j+11$	$j \geq 2$	$6j+5$

The square grid uses $x = i, y = i$ for $-u \leq i \leq u$ and $y = \pm x$. On each grid line the two diagonal crossings already lie on grid lines. On the diagonals the integer coordinates also lie in $[-u, u]$. Including infinity gives at most $2u+2$.

For the full triangular grid let $K = 2k$ and restore all omitted indices in S_K. On an axis or diagonal with $|i| \leq K-2$, the interval calculation in Lemma 3 gives at most $3K+1$ vertices (include the possible half-integer according to parity). At $|i| = K-1, K$ the same count is at most $3K+2$. The medians have at most $3K+2$ by the even–odd count using $[-K, K]$ now. This proves the second row.

For the third row remove $X : x = 3$, $Y : y = 3$, $Z : x + y = 3$ and $H : y = -3$. The following ordinary intersections cover all lines that might exceed $3K+1$. For each $d \in \{K-1, K\}$ the table gives a removed partner:

L	$x=d$	$x=-d$	$y=d$	$y=-d$	$x+y=d$	$x+y=-d$	$x=y$	$x+2y=0$	$2x+y=0$
partner	Y	H	X	Z	H	X	Z	X	Y

For example, $(d, 3)$ has $d+3 > K$, and $d > 3$ excludes all three medians. In the other five grid columns the unused grid coordinate or coordinate sum also exceeds K in absolute value. The three median columns have a half-integral coordinate (in the last case $x = -3/2$), excluding the other grid pencils. Direct substitution excludes the remaining medians. Deleting infinity keeps nonconcurrence in each construction, since two parallel lines and a transverse line survive.

The eight omitted sizes have the following exact witness values:

n	10	11	14	15	22	23	26	27
$t(P)$	5	5	6	6	11	10	13	13

The 10- and 22-line witnesses are triangular grids without medians. The 26- and 27-line witnesses are deletions of the full grid at $K = 4$. For $11, 14, 15$ the coordinate field is $\mathbb{Q}(\sqrt{5})$, and for 23 it is $\mathbb{Q}(\sqrt{2})$. The coordinate lists and exact determinant test are supplied in the checker. The four progressions and the eight entries exhaust $n \geq 8$ and prove Theorem 1.

3.3 A bound for infinitely many pencil offsets

On a target line $x + q_0 y = c$, a pencil $px + qy = k$ with $d = q - q_0 p \neq 0$ meets it at $y = (k - pc)/d$. By reflection and exchange of axes it suffices to take $c \geq 0$ on each of the two axis bands, one diagonal and one knight direction. Splitting s by parity makes each band width $B_1 t + B_0$ with a fixed offset parity ρ. Choose Λ divisible by all $|d|$. In the integer coordinate $z = \Lambda y$, the pencil gives a progression of step $2\Lambda/|d|$ with center $h_c c$, radius $h_t t + h_0$, where $h_c = -\Lambda p/d$ and $(h_t, h_0) = \Lambda(B_1, B_0)/|d|$. Its residue is $h_c c + \Lambda\rho/d$ modulo that step.

Put $M = 2\Lambda$. For an active residue $\sigma \pmod M$, let L_σ be the smallest of the contributing lower endpoints and U_σ the largest upper endpoint. Its occupied heights are contained in this interval, whether or not the progressions leave internal gaps. The total finite count is at most

$$H(c,t) = \sum_{\sigma \text{ active}} \left(\lfloor (U_\sigma - \sigma)/M \rfloor + \lfloor (\sigma - L_\sigma)/M \rfloor + 1 \right). \tag{1}$$

A residue is active if a pencil has its congruence. If a short band contributes no height in that residue, the floors for its bounding interval can count zero.

Lemma 4. *Consider $0 \leq c \leq vt + w$ of fixed parity and $t \geq 0$. Suppose c, t have periods P, Q so that P is even, $P \mid vQ$, and $M \mid h_c P, h_t Q$ for each pencil. On a cell $c \equiv c_0 \pmod P$, $t \equiv t_0 \pmod Q$, it suffices to test $H \leq Nt + N_0$ at the two ends of the first nonempty row, and bound the increase along either end by NQ per step.*

Proof. The divisibilities fix which pencils contribute to each residue. After the floor operation every endpoint gives an affine function of the cell indices. Equation (1) adds maxima of these functions, hence is convex in the offset index. Its maximum on any row is achieved at an end. The two endpoint increments in (c, t) are $(0, Q)$ and (vQ, Q). The increase of a maximum is at most the largest increase of its terms. Summing these bounds proves the claim by induction over rows. □

For instance take an A diagonal at $s = 2t$, with $0 \leq c \leq 16t + 6$, $\Lambda = 6$, $M = P = 12$, $Q = 3$ and target $60t + 23$ finite heights. On the cell $c \equiv 0 \pmod{12}$, $t \equiv 0 \pmod 3$, the active residues are $0, 2, 6, 10$. The first row has $c = 0$ and $H = 13$. The upper bounds on successive boundary increases are 98 and 180, each at most $60 \cdot 3$. At $t = 3$ the five hull values are shown in Figure 3.

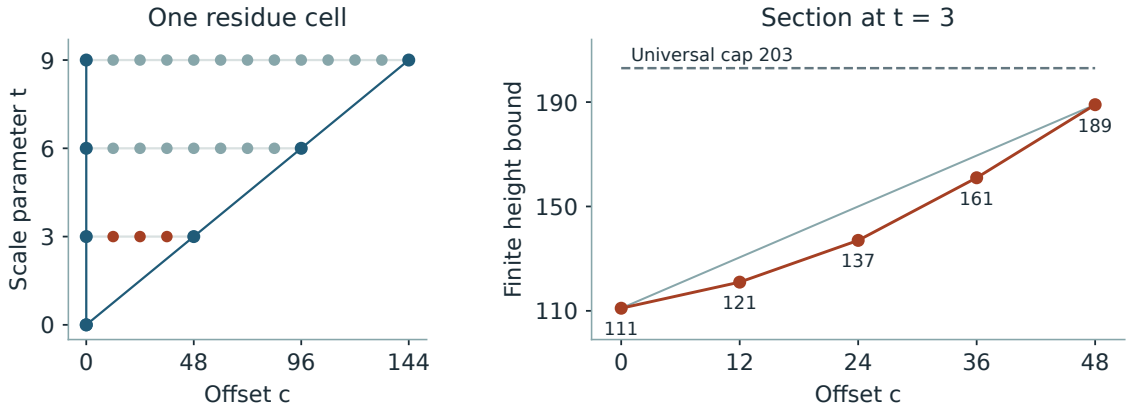


Figure 3. For this residue cell the left and right endpoints advance by 0 and 48 offsets every three steps in t. At right the convex upper bound stays below the chord joining its endpoints and below the target 203. There is one further vertex at infinity.

3.4 Attainment and the effect of removing lines

For A, B there are four target bands, two choices of s parity and two families. The resulting sixteen applications of Lemma 4 bound $r(L)$ for every finite line by $30s + 24$ or $30s + 32$, respectively. The infinity line has just eight vertices. All certificate parameters and every starting and slope comparison are specified in Section 4.

To see equality without an enumeration, write $\delta = 0$ for A and $\delta = 1$ for B. On $x = a$ the horizontals give $I(a)$ and $I(e)$. Additional vertices, with both signs and consecutive terms spaced by two, have positive representatives in the following three intervals:

source	interval	count per sign
$(1, \pm 1)$	$[a + 2, a + b]$	$4s + 3 + \delta$
$(2, \pm 1)$	$[a + b + 2, a + b + 6s + 4 + 2\delta]$	$3s + 2 + \delta$
$(1, \pm 2)$	$[e + 2, 7s + 4 + 2\delta]$	$2s + 1 + \delta$.

Substitution in $y = (c - pa)/q$ realizes each indicated height. The first two intervals lie beyond the outer horizontal of their parity. The last extends the opposite parity beyond $I(e)$. Including infinity gives $30s + 24 + 8\delta$ distinct vertices, attaining the upper bound.

For $C = B \setminus D_8$ twenty additional hull certificates, shared with the other deletions, handle the interior lines. In B they bound $r(L)$ by $30s+28$ at offsets of magnitudes at most $a-2, e-2, b-2, m-6$ in the four bands. At e and $m-4$ they give $30s+30$ and $30s+29$. The table records the compensating ordinary intersections with deleted lines at the boundary. Entries are lower bounds on the number lost.

	$\lvert c\rvert = a, b, m$	$\lvert c\rvert = e$	$\lvert c\rvert = m - 4$
$B \setminus D_8$	4	2	1

One can obtain these integers by intersecting each selected pair and testing the other pencil equations. For example on the outer axis of B the four lines of D_8 with $c = -(m - 2)$ meet it at ordinary vertices. For $c = m - 2$, the two $(1, \pm 2)$ intersections have heights $\pm(2s + 3)$ in $I(a) \cup I(e)$, and the other two lie on diagonals with offset $-4s - 6$ in $I(b)$. Thus exactly four vertices disappear from $x = a$, giving equality at $30s + 28$.

For $B \setminus D_4$ one uses at least one lost vertex on the maximal boundary lines, including the surviving second-outermost knight directions. For A, $s \geq 2$, the interior bounds are already $30s + 21$. At each of a, e, b, m, at least three ordinary vertices are lost upon deleting D_8, and at least one upon deleting D_4. The surviving second-outermost knights in $A \setminus D_4$ also lose an ordinary vertex. These finite boundary tests are integer determinant inequalities in s, checked in the formal proof. Their verification uses the indicated lower limits on s. At $s = 0$ the 61-line deletion has maximum 29 rather than the formula's 28.

Finally, in every row remove the diagonal $x + y = b$. Deletion cannot raise any $r(L)$. In each of the first three families the extremal line $x = a$ keeps every vertex: the intersection with the removed line has height $b - a = -s - 1$, which belongs to $I(a)$ and still lies on the horizontal $y = b - a$. The analogous lower-bound argument on the outer knight is given in Section 4. The infinity line and both axis directions survive. Since $n \equiv 1 \pmod 4$, $G(n - 1) = G(n)$. Together these arguments prove both sizes in each row of Theorem 2.

4 Computational verification and exact attainment

The sixteen baseline certificates use 2,984 parameter cells. Twenty certificates for the surviving interiors use 4,432 more. Their input is a family, a target line direction and parity, Λ, the affine offset cap, the two periods, and the desired affine count bound. Replaying (1) on all 7,416 cells proves every starting and step inequality. Lowering the outer-axis intercept in any of the four baseline parity/family branches by one produces a failing certificate.

For the boundary deletions there is a separate all-scale test. Write the intersection of a surviving line and a candidate ordinary partner in coordinates with fixed positive denominator d. A third pencil can pass through that vertex precisely when its candidate offset $(us+v)/d$ lies in its parity band $I(ws+z)$. This requires two affine inequalities and $us+v \equiv d(ws+z) \pmod{2d}$. Intersect the two integer half-lines with the allowed range of s and test at most $2d$ consecutive integers from its left endpoint. Empty feasibility for each third direction proves the intersection ordinary for every s in the row. This algorithm supplies all boundary losses used above.

An exact count on the outer knight. For each of the last three rows of Theorem 2, take the line $x+2y=m$. Its vertex count can be determined by keeping the full union in each residue instead of its hull. A deleted knight pencil contributes the two offsets $\{-m, m\}$ and the middle interval $I(m-4)$. On the target use $z=60y$ and group by z mod 120. The nonzero determinants divide 60. On a residue σ, a pencil piece with scaled endpoints L, U contributes the half-open interval of quotient indices

$$\big[\lceil (L-\sigma)/120 \rceil,\ \lfloor (U-\sigma)/120 \rfloor + 1\big).$$

Sum the lengths of the unions of these integer intervals and add the vertex at infinity. This counts every occupied height once, including overlaps between surviving pencils.

Only residues compatible with the pencil progression modulo $2\Lambda/|d|$ contribute its interval. Here is a finite test of the resulting equality for all s. Put $s = s_0 + 12u$ with s_0 among the first 12 admissible parameters. Every scaled endpoint changes by a multiple of 120 over this period. All quotient endpoints have the form $\alpha u + \beta$ with integral coefficients. For each residue the order of its endpoints becomes permanent after their last crossing. An upper bound for the first permanent row is

$$u_* = \max\big(\{0\} \cup \{1 + \lfloor (\beta' - \beta)/(\alpha - \alpha') \rfloor : \alpha \neq \alpha'\}\big)\,,$$

taken over pairs of endpoints in the same height residue. After u_* every interval union length is affine, because its occupied components have fixed endpoint labels. Check rows 0 through $u_* + 1$. In the three families $u_* \leq 1$ for every starting residue. The exact counts are, respectively, $30s + 31, 30s+21, 30s+23$. This proves the three remaining lower bounds. Their selected diagonal deletion preserves the maximizing knight: the intersection with $x + y = b$ has height $m - b = -3s - 3$ in $I(a)$. The surviving horizontal through it keeps that vertex. These equalities supplement the upper bounds verified in Lean.

The independent finite-instance calculation canonicalizes pairwise cross products by primitive integer coefficients. It checks 28 maximizers in $A(0)$, both endpoint signs of the punctured grid and all small exceptions. The quadratic cases group a point's joins by vanishing determinants in the basis $1, \alpha$, with $\alpha^2 = \alpha + 1$ or $\alpha^2 = 2$.

4.1 The boundary calculation behind the design

Let a, e, b, m now denote the positive leading coefficients of widths that grow along a common scale, with $e < m < b < a$. The integer offsets satisfy $a_{\text{int}} \equiv m_{\text{int}} \not\equiv e_{\text{int}} \pmod 2$ and b_{int} is even. Write $H_0 = a + e + b + 2m$ for the leading coefficient of half the size. For each of the four outer lines let R be the leading coefficient of its vertex count. Three changes of interval overlap matter:

$$h = (a - 2e - m)_+, \qquad j = (a - b - m)_+, \qquad k = (b - 2m)_+, \qquad u_+ = \max(u, 0).$$

The four exact boundary excesses are

outer line	$R - H_0$ when $h = j = k = 0$	correction elsewhere
$x = a$	$B_1/2$	$-h/2 - j$
$x = e$	B_2	h
$x + y = b$	$B_3/3$	$j - k/3$
$x + 2y = m$	$B_4/60$	$h/4 + j/2 + k/3$.

These entries can be computed directly from the intervals used in Section 4. More explicitly, for target $x + q_0 y = c$ use $\Lambda = 2, 2, 6, 60$ in the four rows. A source band of leading width w contributes the interval with endpoints $f(-w - pc)$ and $f(w - pc)$, where $f = \Lambda/(q - q_0 p)$. In every compatible residue take the union of these intervals, sum their lengths and divide by 2Λ. Rounding their endpoints changes only a bounded number of vertices before scaling. This finite expression also works when intervals are disjoint. For example on $x = a$ the a-parity ordinates occupy ranges with combined leading count $2a + m - j$. The other parity gives $(a + m - h)/2$. Their sum yields the first row. For an exact check over all real widths, normalize $a = 1$. Partition the tetrahedron $0 \le e \le m \le b \le 1$ by comparisons between endpoints in each residue and by the three gap planes. Both the union length and the displayed expression are affine on every cell. Intersecting triples of walls and testing the vertices gives $18, 16, 20, 23$ tests for the four targets in each of two parity phases, 154 in total. At $(41, 4, 20, 11)$ it gives $98, 80, 79, 459/5$; dropping the gap terms would already give 119 at the outer axis.

Multiplying the four excesses by $27, 9, 24, 30$ and adding gives $3h + 12j + 2k$. If every outer line has leading count at most H_0, the three gaps must therefore vanish. The four inequalities reduce to $B_i \le 0$, which forced the ray $(9, 3, 8, 5)$ in Section 2. This conclusion concerns leading widths in the specified pencil model. The finite intercepts in Theorem 2 require the separate height and deletion proofs.

5 Formal verification in Lean 4

The `StrongDirac` library, pinned at `18e92d3c`, proves Theorem 1, the three exact odd-family maxima for A, B, C, and all twelve upper bounds of Theorem 2 for sets of noncollinear points in $\mathbb{R}^2$. The formal proof establishes both the soundness of the integer certificate tests and the truth of their concrete instances by kernel evaluation. Its audit covers 198 declarations in seven source files, with zero unfinished proofs and only `propext`, `Classical.choice` and `Quot.sound`. The lower bounds from the outer-knight interval unions and the retained-horizontal arguments for the even maxima are proved here. The accompanying Python program verifies the exact unions independently of the Lean upper bounds.

6 Prior work and open questions

Akiyama, Ito, Kobayashi and Nakamura formulated the all-size problem; their conference paper gives the earlier constructions and the journal abstract identifies the remaining progression $12k+11$, $k \geq 4$ [1, 2]. Grünbaum discusses Felsner's earlier family with odd size $12k+7$ and maximum at most $6k+2$. His own proposed eventual values, stated by residues modulo 12, are exactly $2\lfloor n/4 \rfloor$ [4]. Florek proves $t(P) \geq \lceil n/2 \rceil$ when the points lie on three concurrent lines [3]. Lund, Purdy and Smith construct $18j+7$ pseudolines with at most $8j+2$ vertices on each; their growing deficit uses the freedom to bend the curves [5]. The constructions here use straight lines throughout.

We consulted the primary conference version, the journal abstract, and the full author versions of Grünbaum, Florek, and Lund–Purdy–Smith. The rest of the Akiyama et al. journal article awaits comparison. The results place upper bounds on $t(n)$; they leave the optimal value at these sizes undetermined. Can a straight-line configuration achieve a deficit greater than $5/2$ along an infinite sequence of odd sizes? Can the deficits grow without bound? A negative answer to the latter would establish the bounded-deficit form of Strong Dirac.

7 Data, reproducibility and use of AI

Running `python3 checker.zip` beside this paper replays the displayed Python example, the finite configurations, all hull and boundary certificates, and the outer-knight interval unions. The archive contains the full verifier and its inputs and uses the Python standard library. The source directory contains the numerical output, printable vector figures, figure-generation code, complete proof-source locations and bibliographic checks. AI systems assisted with the constructions, proof formalization, manuscript preparation and computational checking. Human mathematical and bibliographic review remains part of preparation for submission.

References

[1] J. Akiyama, H. Ito, M. Kobayashi and G. Nakamura, Arrangements of n points whose incident-line-numbers are at most $n/2$, JCCGG conference paper (2009).

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[4] B. Grünbaum, Dirac's conjecture concerning high-incidence elements in aggregates, *Geombinatorics* **20** (2010), 48–55.

[5] B. Lund, G. B. Purdy and J. W. Smith, A pseudoline counterexample to the strong Dirac conjecture, *Electron. J. Combin.* **21** (2014), P2.31.