Two infinite families that avoid palindromes under base-six reverse-and-add

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Abstract

In base six, repeatedly adding a number to its digit reversal can avoid palindromes forever. We give two explicit infinite families with this property. Each number has a fixed middle word flanked by a run of fives and an equally long run of zeros. After twelve additions the same word returns with three new digits in each run. The proof depends on a local fact about addition. A column whose two digits sum to five transmits its incoming carry unchanged, however many times that column is repeated. This reduces an identity for every run length to a computation on three finite blocks. The two middle words are digitwise complements. The resulting families and their iterates fall into six disjoint trajectories, which supply exactly $12(D-110)$ witnesses below 6^D for $D \geq 113$. A finite-state search found the first middle word. The return identity, palindrome avoidance and count each have a proof in the Lean kernel independent of that search. Earlier work supplied the two-run search method and compatible exterior patterns. The middle word completes those patterns and gives a base-six existence proof. The smallest listed base-six candidate remains unresolved.

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1 The two words

Start with the digit string 14 in base six, reverse it and add. The sum $14_6 + 41_6 = 55_6$ is already a palindrome. Does every positive integer eventually reach one? Let $T(n) = n + \mathrm{rev}_6(n)$, where reversal acts on the usual base-six expansion. Call n *Lychrel in base six* if none of $n, T(n), T^2(n), \ldots$ is a palindrome. The following construction answers the question by exhibiting a pattern that persists as the numbers grow.

Read strings from left to right, most significant digit first. Write $d^{[m]}$ for m copies of the digit d and $\bar{d} = 5 - d$ for its complement. Let M be the 98-digit word given exactly in the Python example below, and $\overline{M}$ its digitwise complement. For every integer $m \geq 0$, define x_m, y_m by their base-six expansions

$$x_m = \big(\texttt{133535}\ 5^{[m]}\ M\ 0^{[m]}\ \texttt{003443}\big)_6, \tag{1}$$

$$y_m = \big(\texttt{13343}\ 0^{[m]}\ \overline{M}\ 5^{[m]}\ \texttt{52443}\big)_6. \tag{2}$$

Spaces mark block boundaries. In particular, a zero at the start of a block occupies a column even when it is the first digit of $\overline{M}$.

Theorem 1. *For every $m \geq 0$, $T^{12}(x_m) = x_{m+3}$ and $T^{12}(y_m) = y_{m+3}$. All intermediate words, including the starting word, are nonpalindromic. Thus every x_m and y_m is Lychrel in base six.*

Theorem 2 will also show that the iterates of $x_0, x_1, x_2, y_0, y_1, y_2$ are pairwise distinct, giving exactly $12(D-110)$ of these numbers below 6^D for $D \geq 113$. Both proofs use the same twelve additions. For the return we track three carries per step. For disjointness we also read a handful of digits from those steps.

Here is the first return at $m = 0$. The two quoted pieces of `M` join end to end. The last assertion checks the location and value of every digit after the twelfth addition.

```
M = ('5120533331114012031402333345503345444504511114303'
     '03034021054104350424155334331541311103221333350 11')
def b6(n):
    return str(n) if n < 6 else b6(n // 6) + str(n % 6)
seed = word = '133535' + M + '003443'
for _ in range(12):
    assert word != word[::-1]
    word = b6(int(word, 6) + int(word[::-1], 6))
assert word == '133535' + '555' + M + '000' + '003443'
```

The same pattern, six digits longer

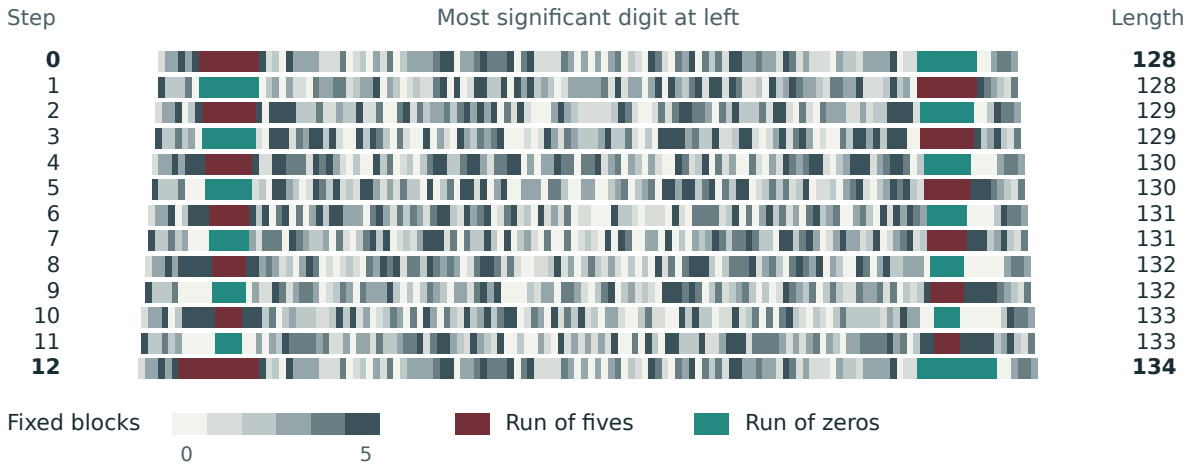


Figure 1: One tile per base-six digit, from x_9 to x_{12}. The colored strips mark the two designated runs. Intermediate rows use the block boundaries of the proof. The last row uses the original boundaries again. The first and last gray centers agree digit for digit. The search registered 287,670,768 distinct center states in finding this word. The proof below checks a twelve-step identity for arbitrary m. The status of the smaller candidate 4555_6 is open.

2 What a run does to a carry

In a base-six addition, the incoming carry c is either zero or one. If two facing digits sum to five, then

$$5 + c = \begin{cases} 6 \cdot 0 + 5 & c = 0, \\ 6 \cdot 1 + 0 & c = 1. \end{cases}$$

The outgoing carry equals c and the written digit is $5(1 - c)$. Replace one such column by any number of identical columns. The next column receives exactly the same carry. This is what makes the parameter m disappear from the block calculation.

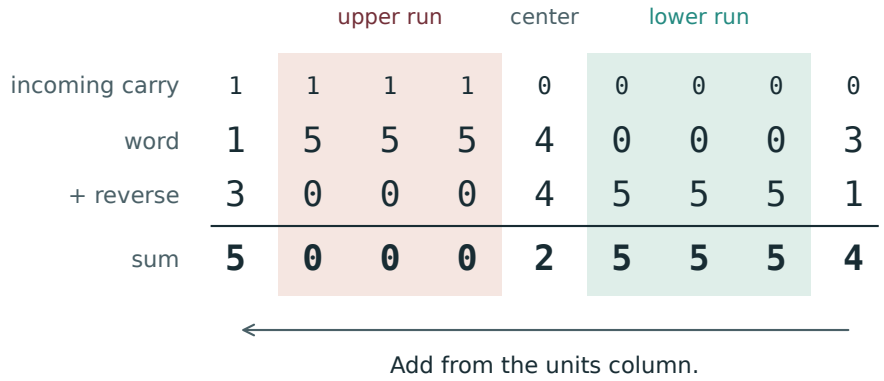


Figure 2: An addition by hand. Each highlighted run pairs three fives with three zeros. The units column supplies a zero carry to the lower run. The central sum $4 + 4 = 12_6$ sets the carry to one, which crosses the upper run. Inserting another column in each run preserves those two boundary carries.

For the general step, let $F = (A, h, N, l, Z)$ denote the word $F[k] = A\, h^{[k]}\, N\, l^{[k]}\, Z$, with A, Z of the same width, A starting in a nonzero digit, and $h + l = 5$. The letter N denotes the current center, which changes during a period. A nonempty run puts h and l in mirror positions. Their sum is odd, so $F[k]$ differs from its reversal whenever $k \geq 1$.

Let R reverse a block at its written width. To compute $T(F[k])$, add three pairs of blocks, retaining leading zeros in each output. In the following display each arrow gives the new block and its outgoing carry, and the new block keeps its input width.

$$\begin{aligned} Z + R(A) &\longrightarrow (Z', a), \\ N + R(N) + a &\longrightarrow (N', b), \\ A + R(Z) + b &\longrightarrow (A', c). \end{aligned}$$

In formulas, if the width is w and the sum is s, the block represents $s \bmod 6^w$ and the carry is $\lfloor s/6^w \rfloor$. The lower run produces $l' = 5(1 - a)$ and the upper run $h' = 5(1 - b)$. Regrouping the result gives

$$\begin{aligned} c = 0: &\quad (A', h', N', l', Z')[k], \\ c = 1: &\quad (1A',\ h',\ h'N',\ l',\ l'Z')[k - 1] \quad (k \geq 1). \end{aligned} \tag{3}$$

For $c = 1$ the most significant carry prefixes the digit 1. Moving one upper-run digit into the center and one lower-run digit into the right block restores equal end widths. The remaining symbolic run length drops by one, although the integer itself has gained a digit. The next row is again a form of the required kind precisely when the end and digit conditions hold. In our rows the new runs are complementary because $a + b = 1$.

3 The finite return certificate

The three block sums are a recurrence on short strings, so a reader can check the full certificate using the two words in (1)–(2). Table 1 gives the carries for all twelve steps and the resulting fixed blocks. Here a step is numbered by the row it leaves, starting with row zero. The rightmost bit in each triple is the overflow c.

Table 1: Twelve applications of (3). Within each row parity the same carry triple recurs six times. The symbols A, h, N, l, Z in the last line denote the respective family's initial blocks.

	x	y
(a, b, c) at $t = 0, 2, 4, 6, 8, 10$	$(0, 1, 0)$	$(1, 0, 0)$
(a, b, c) at $t = 1, 3, 5, 7, 9, 11$	$(1, 0, 1)$	$(0, 1, 1)$
Initial left and right blocks	`133535, 003443`	`13343, 52443`
Left and right after one step	`522240, 543214`	`52212, 31214`
Both final forms	$(Ah^{[6]},\ h,\ h^{[3]}Nl^{[3]},\ l,\ l^{[6]}Z)$	

Proof of Theorem 1. Start with $m \geq 6$. After t steps the remaining run length is $k = m - \lfloor t/2 \rfloor$, by the overflow column of Table 1. For $0 \leq t < 12$ this is at least one. The opposite run digits therefore rule out a palindrome at every one of these phases. At the endpoint $k = m - 6$ may be zero. On the upper side the last line of the table places 6 copies of h at the end of the left block, then $m - 6$ in the remaining run, then 3 at the start of the center. These merge into a run of length $m + 3$. The three groups of l on the lower side give the same sum in reverse order. The outer blocks and the original middle word now have exactly their starting values.

For $m = 0, \ldots, 5$, direct integer addition on the two explicit templates gives the same return, the same overflow bits and no palindrome at $t = 0, \ldots, 11$. These twelve finite cases are checked both by `recompute.py` and by reduction of the digit lists inside Lean. Finally write any iteration time as $12q + r$ with $0 \leq r < 12$. The r-th iterate from parameter $m + 3q$ is nonpalindromic by the single-period argument. This includes the original seed and every later iterate. □

Why the second center is free. Consider a base $B \geq 2$. Write the sum of one column as $a + b + c = Bk + d$, with incoming carry c, outgoing carry k, and output digit d. Then

$$(B - 1 - a) + (B - 1 - b) + (1 - c) = B(1 - k) + (B - 1 - d).$$

Thus complementing both inputs and the incoming carry also complements the output and the outgoing carry. In base six the x and y columns enter their centers with opposite carries

and leave their outer blocks with the same overflow bit. Complementing the center at one phase therefore complements it at the next. This also holds on an overflow, when the new upper-run digit moves into the center.

More generally, the center search maintains boundary carries for all phases and buffers digits that an overflow will shift one position. Map each carry to $1 - c$ and each stored or guessed digit to $B-1-d$. The column identity preserves every transition, including a midpoint column where the two inputs coincide. The two closing carries agree before the map exactly when they agree after it. Since the map is its own inverse, this gives a bijection of accepting center paths for opposite incoming carry signals. An overflow always appends a leading 1, so the outer blocks require a separate solution.

4 Six trajectories and a count

Write f for either x or y. Since T^{12} adds three to the index, every f_m belongs to the trajectory from f_r for $r = m \bmod 3$. It remains possible in principle for two trajectories to merge between returns. Excluding such a merger gives an exact count.

Theorem 2. *All forward iterates of $x_0, x_1, x_2, y_0, y_1, y_2$ are distinct, even across seeds. For $D \geq 113$, exactly $12(D - 110)$ of these iterates are below 6^D.*

Proof. Let L_f be 110 for x and 108 for y. The alternating overflows in each period, including the finite small cases, give

$$\ell(T^t(f_m)) = L_f + 2m + \lfloor t/2 \rfloor,$$

where ℓ counts digits. Suppose two words on the six trajectories are equal. Advance both until the first is at phase zero and both indices at the beginnings of their periods are at least six. This is possible since each full period adds three. In a cross-family comparison take the x trajectory first. Its fourth digit at phase zero is 5. Every y phase has fourth digit 4 (even phase) or 1 (odd phase). These values follow from the outer block recurrence (3).

For a within-family comparison let the other phase be u, $0 \leq u < 12$. Its digit length must be even. This excludes $u = 2, 3, 6, 7, 10, 11$. Of the remaining phases $1, 5, 9$ begin with 5, whereas phase zero begins with 1. At $u = 4$ or 8 the center distinguishes the words. To see which digits line up, put $j = u/4$. If the phase-u trajectory began its period at index n, its active run length is $k = n - 2j$ and its left block has grown by $2j$ columns. Equal lengths force the phase-zero parameter to be $n + j$. Thus the first center digit at phase zero lines up with center digit j (counting from zero) at phase u. The three digits from the recurrence are

	$M^{(0)}[0]$	$M^{(4)}[1]$	$M^{(8)}[2]$
x	5	1	3
y	0	4	2

and in either family both comparisons fail. Only phase zero remains. The common length fixes the parameter and hence its residue modulo three, so the seeds agree. For a positive integer the reversed value is positive, so the orbit is strictly increasing and the times must also agree.

A seed of length L supplies $2\max(0, D + 1 - L)$ iterates with at most D digits. The six initial lengths are $110, 112, 114, 108, 110, 112$, whose sum is 666. At $D = 113$ the term

for length 114 is already zero without truncation. For all $D \geq 113$ the total is therefore $12(D+1) - 2 \cdot 666 = 12(D-110)$. Each of these integers is Lychrel by Theorem 1. □

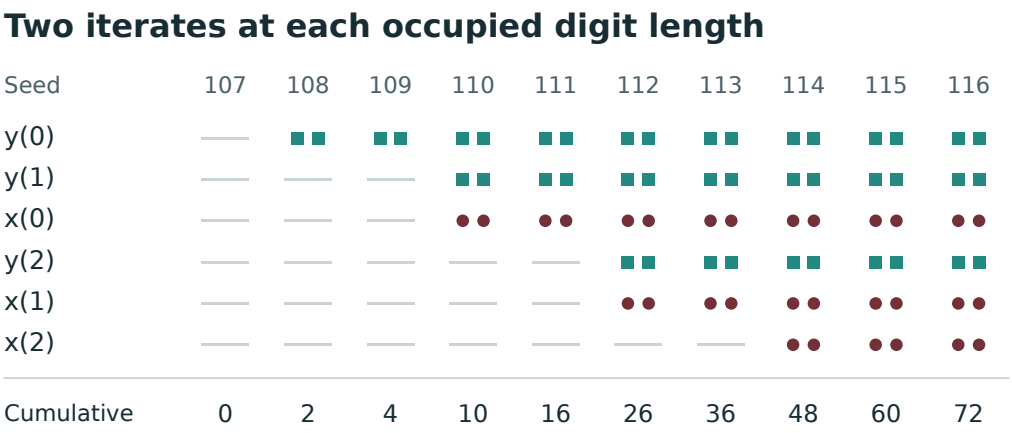


Figure 3: Columns correspond to exact base-six digit lengths. Each symbol is a distinct integer, computed by ordinary reverse-and-add from the indicated seed. The last row counts all six trajectories up to and including the column length. All six contribute at length 114.

5 Finding the middle word

An ordinary orbit computation can suggest a pattern only after a seed is known. Here the search worked in the opposite order. It prescribed which of the twelve additions would overflow and sought digits realizing that prescription. The two ends of the word can be extended in mirror pairs. Their boundary carries prescribe the signals that a compatible middle word must accept. A breadth-first search for the middle then adds one mirror pair at a time, enforcing all twelve additions simultaneously.

Represent an overflow schedule by a binary word up to cyclic shift (a *necklace*). Among the 179 positive even-weight necklaces of length 12, the alternating schedule was the only survivor in the search for equal-length runs of zeros and fives in base six. The search exhausted the other 178 schedules with no accepting path. The outer calculation gave eight carry signals. Six exhausted their compatible centers without a solution. Kusayama's search on the other two had reached its cap of 2^{24} states [4]. The extended search for one signal registered 287,670,768 distinct states before stopping with an accepting path of 49 mirror pairs. Reading those pairs gives M. For the other signal the complement bijection gives $\overline{M}$ directly. The search supplied candidate words. The digit calculation in Section 2 proves what they do.

The extent of the search matters for exclusions. In base six the enumeration also reported no accepting paths for all 220 positive even-weight necklaces of periods 2 through 11 in the tested zero/five two-run class. This statement relies on the recorded C searches and, for the earliest runs, on the assumption that the executable matched its logged source. A state cap leaves a branch undecided. The existence and count in Theorems 1 and 2 require neither search completeness nor the rejection of any other pattern.

A short obstruction in other bases. Write $\omega = B - 1$. Brown's four-step pattern [2] has successive words

$$c\,\omega^{[n]}\,b\,0^{[n]}\,a, \quad f\,0^{[n]}\,e\,\omega^{[n]}\,d,$$
$$c\,\omega^{[n]}\,g\,0^{[n]}\,a, \quad f\,0^{[n]}\,h\,\omega^{[n]}\,d,$$

read across the rows, with the first word returning at $n + 1$. The four exterior words have equal width. For $B \equiv 1 \pmod 3$ the center precludes this pattern. Indeed, writing z' for reversal at the width of the block, addition from the right gives the necessary equations

$$b + b' = e + B^k, \qquad e + e' + 1 = g - (B-1)B^k,$$
$$g + g' = h + B^{k+1}, \quad h + h' + 1 = bB.$$

Here b, e have width k and g, h width $k + 1$. The terms B^k and B^{k+1} account for carries out of those widths. Reversal preserves digit sum, so $z' \equiv z \pmod 3$. Substitution successively yields $e \equiv -b - 1$, $g \equiv b + 2$, $h \equiv -b$ and then $b \equiv b + 1$. This rules out Brown's shape in base ten, where the existence of even a single Lychrel number remains open.

6 Formal verification

The Lean 4 library `LychrelBase6` starts with natural numbers and their canonical base-six digits. It proves that the padded, columnwise addition of reversed words computes the same T used in the theorems. It derives (3) for an arbitrary run length, checks the exact terminal blocks and the six small parameters for each family by kernel reduction, and proves the six-trajectory injection and count. Its general base-B result concerns the center automaton, including paths and both midpoint parities. The audit inspected proof revision `9c494830` and is recorded at `c052ec65`. It found only standard logical axioms in 155 declarations and no `sorry` or `native_decide`. The large state counts come from the external search.

7 Earlier constructions and open questions

Digit words that reproduce after reverse-and-add have a long history. Trigg studied the binary trajectories [1], and Brown exhibited self-similar constructions in several bases [2]. A 2016 report and Haskell program already searched the exterior and interior of a two-run word separately [3]. Its implementation allowed at most two overflows. The present word uses six. Kusayama treated general even overflow weights and published both a base-nine family and the capped base-six searches [4].

Two prior results bear directly on the base-six attribution. Kuitché's identities for sibling trajectories in small bases are conditional. The explicit construction in that work specializes to base four [5, 6]. Independently, publicly recorded base-six tests already contain the outer pair-sum sequences of both families here. Those tests give the exterior only [7]. The new ingredient relative to those records is a compatible center yielding a proof of base-six existence. The literature examined gives no earlier such center. Historical priority across unindexed and inaccessible records remains open.

OEIS A060382 lists $1079 = 4555_6$ as the smallest apparent base-six candidate [8]. Its first iterate is 14553_6. The two families above leave its eventual behavior open. Does it ever enter

one of the six trajectories? Is there a smaller explicit word with a provably nonpalindromic orbit? The count here grows only linearly in the digit length. Determining whether other orbits give substantially faster growth would distinguish scarcity of these templates from scarcity of Lychrel numbers themselves.

8 Reproducibility and authorship

The accompanying `recompute.py` reads and runs the Python example in Section 1, evaluates each intermediate block, and compares the symbolic calculation with ordinary integer addition. The figures are generated by `figures/make_figures.py` from the same starting words. The directory `lean/LychrelBase6` contains the formal proof and axiom checks. Source and literature ledgers accompany the manuscript. AI systems assisted with the search, formalization, exposition, and figures. Human review and authorship approval are pending.

References

[1] C. W. Trigg, Versum sequences in the binary system, *Pacific J. Math.* **47** (1973), 263–275. https://msp.org/pjm/1973/47-1/pjm-v47-n1-p25-p.pdf.

[2] K. S. Brown, *Self-similar reverse-sum sequences*, MathPages, https://www.mathpages.com/home/kmath312/kmath312.htm.

[3] Anonymous, *Search for and classification of Lychrel numbers*, report and Haskell implementation (2016), https://www.p196.org/files/lychrel842.zip.

[4] Y. Kusayama, *Self-similar reverse-and-add cycles and a base-9 Lychrel family*, code, tables and preprint (2026), https://doi.org/10.5281/zenodo.23289049 and https://github.com/dew-web/lychrel-self-similar.

[5] R. S. Kuitché, *On the construction of some Lychrel numbers*, Research Square (2023), https://doi.org/10.21203/rs.3.rs-2659802/v1.

[6] R. S. Kuitché, Lychrel numbers in base 10: A probabilistic approach, *Adv. Pure Math.* **14** (2024), 667–694. https://doi.org/10.4236/apm.2024.148037.

[7] taco-jpg/p196-algorithm, Base-six exterior test cases, pull requests 417 and 436 (2026), https://github.com/taco-jpg/p196-algorithm/pull/417 and https://github.com/taco-jpg/p196-algorithm/pull/436.

[8] OEIS Foundation, A060382, *The On-Line Encyclopedia of Integer Sequences*, https://oeis.org/A060382.