Discrete dynamics · Arithmetic iteration

The last-digit Collatz family across all bases

Finite stopping times and divergent residue classes

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Third draft · November 2026

AI-assisted research draft. Prepared for author and independent mathematical review.

Abstract

At an odd integer, multiply by its last base-b digit and add one. At an even integer, divide by two. How does changing the base change the orbit? For every even base we classify the probability of staying above the starting value. At each finite horizon this probability is an exact natural density. Its limit vanishes precisely in bases $2, 4, 6, 10$, giving density-one descent in those four bases. For every other even base the finite-horizon densities stay bounded away from zero. For odd bases a zero digit sends an integer directly to one. Tracking the residue and a finite number of binary digits can also certify that an entire progression diverges. For 905 of the 999 odd bases through 1,999, these two outcomes account for density one. In base 17 both have positive density and a separate periodic orbit still exists. For the decimal map we sharpen the growth exponents for the odd basin of one and bound the size of any further cycle above a stated cutoff. The proofs use established parity coding and Markov methods. Lean 4 checks the structural arithmetic; an accompanying integer verifier replays the finite surveys, residue inequalities and cycle enumeration.

MSC 2020 11B83, 11Y16, 37P99, 60J10. **Keywords** Generalized Collatz maps, stopping time, natural density, Markov chains, inverse iteration.

1 One integer, two bases

Write $d_b(n) = n \bmod b$ for the least nonnegative residue. On positive integers define

$$f_b(n) = \begin{cases} n/2, & n \text{ even}, \\ d_b(n)n + 1, & n \text{ odd}. \end{cases}$$

Start at 33. In base 5 its digit is 3. The orbit is $33, 100, 50, 25, 1$; at 25 the digit is zero. In base 17 the digit of 33 is 16. After ten steps the orbit returns to the same odd residue at 370,871. One can certify everything that happens next without computing another term. Every $n \equiv 33 \pmod{34}$ obeys

$$f_{17}^{10}(n) = 11{,}200n + 1{,}271 \equiv 33 \pmod{34}. \tag{1}$$

Each pass through this word increases the starting value, and the finitely many intermediate affine maps have positive slope. Hence every term tends to infinity along this progression.

Check the whole progression. Write the current value as $aq + c$ for arbitrary $q \geq 0$. A zero in the word below prescribes a halving. Each assertion checks the branch on every q simultaneously.

```
a, c = 34, 33
for d in (16, 2, 5, 0, 0, 0, 0, 8, 14, 10):
    if d:
        assert a % 34 == 0 and c % 2 == 1 and c % 17 == d
        a, c = d*a, d*c + 1
    else:
        assert a % 2 == c % 2 == 0
        a, c = a//2, c//2
assert (a, c) == (34*11200, 33*11200 + 1271)
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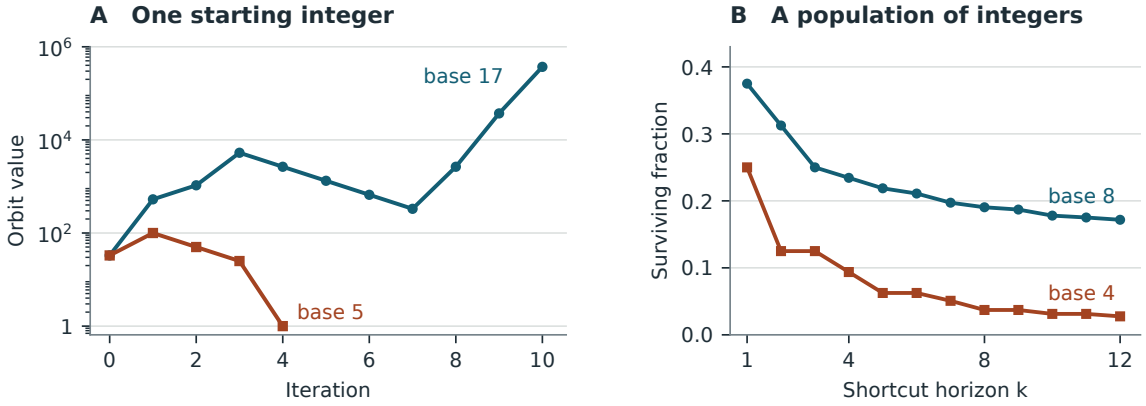


Figure 1: Two routes from orbit data to a theorem. Left, the four consecutive halvings in base 17 lie on the descending segment of the blue trace. Repeating the residue and checking its affine return proves divergence of the entire progression. Right, each point counts all surviving residue classes at that horizon. The drift proof establishes that the limit vanishes in base 4 and stays positive in base 8. The congruence and drift calculations have Lean proofs; the limiting probability uses the finite-state ergodic theorem. Individual integer orbits in base 8 require a separate argument.

For even b, an odd step always produces an even integer. Incorporate its first halving into the *shortcut* T_b: $T_b(n) = n/2$ at an even n and $T_b(n) = (d_b(n)n + 1)/2$ at an odd n. For $k \geq 1$ put

$$D_k(b) = \{n \geq 1 : T_b^j(n) \geq n \ (1 \leq j \leq k)\}, \qquad p_k(b) = d(D_k(b)), \qquad p_\infty(b) = \lim_{k\to\infty} p_k(b).$$

Here $d(S) = \lim_{X\to\infty} |S \cap [1, X]|/X$ when the limit exists. Section 2 proves existence of each p_k. The sequence decreases, so the final limit exists as well.

Theorem 1 (Even bases)**.** *For every even $b \geq 2$,*

$$p_\infty(b) = 0 \quad \Longleftrightarrow \quad b \in \{2, 4, 6, 10\}.$$

In each of these four bases the set of integers whose orbit eventually falls below its starting value has natural density one. In every other even base there is a constant $\eta_b > 0$ such that $p_k(b) \geq \eta_b$ at every finite horizon k.

On the positive side, the order of the two limits matters. A set can have positive density at every finite horizon while the infinite intersection has density zero. For example, the sets $\{n : n \geq k\}$ have density one at every k and empty intersection. An integer-divergence claim needs information of the kind supplied by (1).

Theorem 2 (Odd bases)**.** *Let $b \geq 3$ be odd. Write Z_b for starts which reach a multiple of b, R_b for starts which reach 1, and E_b for starts tending to infinity. There are computable finite-cylinder lower bounds increasing to masses τ_b, ρ_b for Z_b, E_b, respectively. Whenever $\tau_b + \rho_b = 1$,*

$$d(Z_b) = d(R_b) = \tau_b, \qquad d(E_b) = \rho_b.$$

If a prime congruent to $3 \bmod 4$ divides b, then $Z_b = b\mathbb{N}$. For 905 odd bases $3 \leq b \leq 1{,}999$ the certificate establishes $\tau_b + \rho_b = 1$. For 27 of these $\tau_b = 1$; in the remaining 878 both masses are positive.

Theorem 3 (Decimal specialization)**.** *Let $N_1(X)$ count* odd *integers $n \leq X$ reaching 1 under f_{10}. For some constant $C > 0$ and all real $X \geq 1$,*

$$.0806(X/31)^{399/800} \leq N_1(X) \leq CX^{.6245}, \qquad N_1(X) \geq .0145X^{.49875}.$$

The only primitive odd periods of length at most eight are (1) *and* $(5, 13)$*. If a decimal cycle has least odd member at least* 10^{15}*, then its counts* K *of halvings and* m *of odd steps satisfy*

$$K \geq 34{,}905, \qquad m \geq 15{,}496, \qquad K + m \geq 53{,}185.$$

The last statement has a cutoff hypothesis that can be used with the separately recorded exhaustive descent check below 10^{15}. The proof of the short-period statement uses no lower cutoff. The companion decimal paper develops the two basins and the connection with the classical Collatz map [1]. Here the central distinction is between how even and odd bases encode parity. Sections 2 and 3 develop those two arguments. The decimal improvements use inverse moves and a product identity in Section 4.

2 Even bases: when does a typical orbit fall?

2.1 Reading one new bit at a time

For an orbit following a prescribed shortcut word, write

$$T_b^j(n) = \frac{A_j n + B_j}{2^j}, \qquad A_0 = 1,\ B_0 = 0. \tag{2}$$

At an even branch (A_j, B_j) is unchanged; at an odd branch of digit r it becomes $(rA_j, rB_j + 2^j)$. Thus A_j is odd, $B_j \geq 0$, and $A_j \neq 2^j$ for $j > 0$. The sign of $A_j - 2^j$ determines the sign of $T_b^j(n) - n$ for all sufficiently large n in that word.

Lemma 4 (Finite coding)**.** *For* $k \geq 1$*, a class modulo* $b2^{k-1}$ *determines all the branches used by* T_b^j*,* $1 \leq j \leq k$*. The density* $p_k(b)$ *is the fraction of such classes for which* $A_j > 2^j$ *for every* $1 \leq j \leq k$*.*

Proof. If two starts differ by $b2^{k-1}$, their images at time $j < k$ differ by a multiple of b, inductively by (2); the same digit and branch follow. A class passing every product test lies in D_k. If $A_j < 2^j$ at any time, only $n \leq B_j/(2^j - A_j)$ can pass that comparison. There are finitely many classes and times. Discarding their finitely many exceptions gives the asserted fraction. □

The two congruence types of even bases lead to different chains. For $4 \mid b$, an odd n of digit r satisfies $rn + 1 \equiv 2 \bmod 4$. It has exactly one halving, so the shortcut maps odd to odd. The two lifts of an odd residue class of a given word lead with equal frequency to

$$c = (r^2 + 1)/2, \quad c + b/2 \pmod{b}. \tag{3}$$

Indeed a difference $b2^{j-1}$ of two starts at the next refinement shifts the jth image by $A_j b/2$ and A_j is odd. Even initial integers fail the first comparison. Thus p_k is half the probability of passing all product tests in the digit chain started uniformly on odd digits.

For $b = 2M$ with M odd, let $y \in \mathbb{Z}/M\mathbb{Z}$ and let $d(y)$ be its unique odd representative in $[1, 2M)$. A residue modulo $M2^k$ is equivalent to $y_0 = n \bmod M$ and k successive parity bits. To extend a word by one bit, the two lifts differ by $M2^k$, shifting the image at time k by $A_k M$, an odd integer. Every parity extension therefore occurs once. Under uniform counting the bits are independent fair bits. The residue evolves by

$$H(y) = y/2 \pmod{M} \quad \text{on an even bit}, \qquad Q(y) = (y^2 + 1)/2 \pmod{M} \quad \text{on an odd bit}.$$

Only the odd branch multiplies A_j by $d(y)$. At successive odd terms, the number $v \geq 1$ of intervening halvings has probability 2^{-v} and the next residue is $(y^2 + 1)2^{-v} \bmod M$. These formulas include $M = 1$ by using the unique residue modulo one.

2.2 The least drift

In the odd-term chain let π be a stationary law on a closed class and h the mean number of halvings per odd term. The expected logarithmic change of the leading multiplier is

$$\Delta(\pi) = \sum_r \pi(r) \log r - h \log 2, \qquad h = 1 \ (4 \mid b), \quad h = 2 \ (b = 2M, \ M \text{ odd}).$$

Here and below log means natural logarithm. Write Δ_b for the least closed-class drift.

Lemma 5. *If $4 \mid b$, then $\Delta_b = \frac{1}{2}\log(b/2+1) - \log 2$, attained uniquely on $\{1, b/2+1\}$. If $b = 2M$ with M odd, then $\Delta_b \geq \frac{1}{2}\log((M+1)/64)$. In particular $\Delta_b \to +\infty$ through all even bases.*

Proof. For $4 \mid b$, order the two targets of (3) by size. Their product is at least $1(b/2+1)$, with equality exactly when the smaller is 1. From both 1 and $b/2+1$, the targets are precisely that pair; assigning each half the probability gives equality. Any other closed class is disjoint from this pair and has strict inequality in every row. Some transient digits can lead to 1.

For M odd, fix a starting digit r. Set $z = (r^2+1) \bmod M$ and $y_v = z2^{-v} \bmod M$, represented in $[0, M)$. If $z = 0$, the next digit is M. If y_1 is even and positive, the event $v = 1$ alone contributes at least $\frac{1}{2}\log(M+2)$ to the mean of $\log d(y_v)$. If y_1 is odd and $v \geq 2$, the even integer $2^{v-1}y_v$ equals $y_1 + tM$ with $t \geq 1$: a negative t makes it negative, and zero gives the wrong parity. Thus $d(y_v) \geq (M+1)/2^{v-1}$. Since $\sum_{v\geq 2}(v-1)2^{-v} = 1$, each row in all three cases has mean next log digit at least $\frac{1}{2}\log(M+1) - \log 2$. Average in stationarity and subtract $2\log 2$. The lower bound is positive as soon as $M \geq 65$. □

The finite step can be checked without guessing the stationary proportions. Form the exact transition matrix P by summing 2^{-v} over complete periods of 2 mod M, and write $g(r) = \sum_s P(r,s)\log s - h\log 2$. For any stationary law,

$$\Delta(\pi) \geq \min_r \frac{1}{\ell} \sum_{i=0}^{\ell-1} (P^i g)(r).$$

For odd $7 \leq M \leq 63$ this is positive at $\ell = 8$. Rational lower bounds on the logarithms give integer comparisons for this finite test. The exceptional singly even cases $M = 1, 3, 5$ and $b = 4$ each have a single negative class. The same method separates the quarter-unit band: precisely $b = 4, 8, 10, 14$ have a closed class with $|\Delta| \leq 1/4$, and only $4, 10$ have one in $[-1/4, 0)$. For $M \geq 105$ the analytic bound alone exceeds $1/4$; the intervening odd M are covered by the finite certificates. An independent stationarity solve and the enumerated values are described in Section 5.

b	Δ_b	$p_{12}(b)$	b	Δ_b	$p_{12}(b)$
2	−1.386294	0	8	+0.111572	1407/8192
4	−0.143841	113/4096	10	−0.134161	495/4096
6	−0.849815	7/6144	14	+0.067947	5669/28672

Table 1: Odd-term drifts rounded to six places, and exact finite-horizon densities. Base 10 has more survivors at horizon 12 than base 4, although both limiting fractions are zero.

2.3 Passing from the chain back to integers

Proof of Theorem 1. Set $S_j = \log A_j - j\log 2$ in the finite-state shortcut coding. The product test is $S_i > 0$ at all times $1 \leq i \leq k$. For $4 \mid b$ one starts uniformly on odd digits and multiplies the resulting probability by $1/2$. For $b = 2M$, the shortcut kernel is $K = (H+Q)/2$, now writing the two maps as stochastic matrices. The odd-term kernel is $P = \frac{1}{2}Q(I - \frac{1}{2}H)^{-1}$. The equation $\pi P = \pi$ is equivalent, by multiplication on the right, to $\pi K = \pi$. Thus they share the extreme stationary

laws and the shortcut drift under any such law is $\Delta(\pi)/2$, since the next bit is fair. This also holds when the initial state is transient, after entering the eventual closed class.

The finite-state ergodic theorem gives S_j/j tending almost surely to the drift of the eventual class per shortcut step (one can adjoin the next parity bit to make the increment a state function). In the four negative bases $S_j \to -\infty$. The decreasing chain events therefore have limiting probability zero. Every integer never falling below itself lies in each D_k, so its set has upper density at most every p_k and hence zero. Descents occur at the same starting integers for the raw map: its omitted odd-branch intermediate $d_b(n)n + 1$ is larger than n.

In the other bases $S_j \to +\infty$ almost surely. A sample path has a last time attaining its minimum. At some deterministic time l there is positive probability that all later sums are strictly larger than S_l. Conditioning on the state at l shows that one state has a positive chance of strict survival forever from height zero. This state has positive initial probability in the uniform coding. Continuity of probability on decreasing events gives $p_\infty > 0$. Every equality between probabilities and integer densities used here concerned a *fixed* horizon. □

2.4 Quantitative survival

Let m denote the multiplier in one shortcut branch ($m = 1$ on an even bit), and let z, z' be the old and new states. For $t > 0$ and a positive vector w, define

$$(M_t^{\pm} w)(z) = \mathbb{E}_z[(m/2)^{\pm t} w(z')].$$

If $M_t^{+} w \le \lambda w$, then $p_k \le (\sum_z \mu(z)w(z)/\min w)\lambda^k$, where μ denotes the initial mass used for the product test. Indeed $A_k/2^k > 1$ on every surviving path, and the expected weighted tth power is bounded by iteration. In the other direction $M_t^{-} w \le w$ makes $(2^j/A_j)^t w(z_j)$ a nonnegative supermartingale. From a surviving path of height $A_j/2^j = u > 1$, stop on first reaching height at most one, first truncating the stopping time. The chance of later failure is at most $u^{-t} w(z_j)/\min w$. Summing the resulting positive lower bounds over surviving prefixes gives, for example,

$$p_k(6) \le 4.595731(.702852)^k, \qquad p_\infty(8) \ge .0861, \qquad p_\infty(14) \ge .0298.$$

All three decimal inequalities are rounded outward from rational certificates. The first uses $t = 31/16$; the other two sum prefixes of length ten. The download checks all 14 stored vectors.

There is also a bound requiring no per-base vector search. For $b = 4k,\ k \ge 4,\ k \ne 6$, take $t = 1/2$ and set $w(1) = 1$, $w(r) = 2/5$ otherwise. At 1 the ratio bound is $7\sqrt{2}/10 < 1$. At $r \ge 3$, the target pair can contain 1 only when $r^2 \equiv 1 \bmod b$. In that case $r \ge 7$ under these restrictions on b, and $(7/10)\sqrt{2/r} \le 2/5$; otherwise the bound is immediate. Each odd $r \ge 3$ has a successor other than 1 with probability at least half, so the one-step estimate gives

$$p_\infty(b) \ge \frac{1}{2b} \sum_{\substack{3 \le r < b \\ r \text{ odd}}} (1 - \sqrt{2/r}).$$

3 Odd bases: two outcomes certified by a finite word

3.1 What can reach a multiple of the base?

If b is odd, a multiple n of b first halves $v_2(n)$ times to an odd multiple; its next step is 1. Since all preceding terms are still positive multiples of b, this is the first arrival at 1. A nonmultiple entering $b\mathbb{N}$ must be odd, since halving preserves divisibility by odd b. Its digit r must satisfy

$$r^2 \equiv -1 \pmod{b}.$$

A prime factor $p \equiv 3 \pmod 4$ rules this out, so $Z_b = b\mathbb{N}$ exactly in that case. If all primes dividing b are 1 mod 4, each has two roots, each root lifts uniquely to every prime power since $2r$ is a unit,

and the Chinese remainder theorem gives $2^{\omega(b)}$ roots. Here $\omega(b)$ counts distinct prime factors. Each entry root specifies one odd residue modulo $2b$. The density of starts already in $b\mathbb{N}$ or entering it on the first step is

$$\frac{1}{b} + \frac{2^{\omega(b)}}{2b}.$$

Later entries can dominate this immediate fraction. For $b = 5$, consider an odd $n > 1$. Digits $2, 3$ enter a multiple of 5; digit 4 gives the odd integer $4n + 1$ of digit 2; digit 1 gives the even integer $n+1$, then $(n+1)/2 < n$. Together with the zero digit and the even branch this proves by induction that *every* positive start reaches 1.

3.2 Using the low binary bits

For $0 \le x < b$ and $j \ge 0$ define the cylinder

$$\mathcal{C}(x, c, j) = \{n \ge 1 : n \equiv x \bmod b,\ n \equiv c \bmod 2^j\}, \qquad 0 \le c < 2^j.$$

It has density $1/(b2^j)$. When $j = 0$ expose one parity bit, splitting the starting cylinder into two equal halves. When $j \ge 1$ and $x \ne 0$, the following moves require no additional bit:

parity of c	new residue modulo b	new low bits	number known
even	$x/2$	$c/2$	$j - 1$
odd	$x^2 + 1$	$xc + 1 \bmod 2^{j+v_2(x)}$	$j + v_2(x)$.

In the first row $x/2$ denotes modular division. In the second row the current *integer digit* x is multiplied, so its factor of two adds known bits. In either row specifying e further bits of the output specifies exactly e further bits of the input. For division, lift the congruence by a factor of two. For multiplication by x, divide its known power of two out of the congruence and invert the odd part modulo 2^e. Induction now gives a path using s splits from residue x as a single starting cylinder of density $1/(b2^s)$.

A deterministic stretch starts from an odd digit $(x, 1, 1)$ and proceeds while $j \ge 1$ and the residue is nonzero. If it enters an earlier cylinder as a subset, every number in that earlier cylinder repeats the intervening word. Suppose the word has h halvings and its other steps have digits $d_1, \dots, d_l$. Containment forces $h \le \sum_i v_2(d_i)$ by the bit counts. At least one step is an odd multiplication, since only halvings would lose known bits. Thus the return map is $(An + B)/2^h$ with $A = \prod_i d_i \ge 2^h$ and $B > 0$. Successive returns are strictly increasing integers, and each intermediate term is at least the previous return divided by 2^h. This proves divergence of the whole progression. An odd stretch which enters a previously certified divergent cylinder is divergent as well. Equation (1) is an example with $h = 4$.

Proof of Theorem 2. Observe the process at states with zero bits known. At a nonzero residue x, assign probability $1/2$ to halving and returning to $x/2$, and $1/2$ to the deterministic stretch from $(x, 1, 1)$. Its endpoint may be residue zero, a certified escape, or another zero-bit state. A stretch on which no outcome has been proved contributes no absorbed mass. Give residue zero its initial share $1/b$. Let τ_b, ρ_b be the probabilities of certified finite entry into the zero residue and escape classes. All first-hit paths are disjoint; every finite collection is a union of starting cylinders with exactly the indicated natural density. Taking a supremum proves $\underline{d}(Z_b) \ge \tau_b$, $\underline{d}(E_b) \ge \rho_b$. If $\tau_b + \rho_b = 1$, the inclusions $Z_b \subseteq R_b$ and $E_b \cap R_b = \emptyset$ squeeze both upper densities to the same bounds. This proves all three density formulas without interchanging a density limit with a countable union.

If all stretches have been resolved, the reduced graph is finite. Absorption has probability one precisely when each state can reach an absorbing outcome. In that case a path to absorption of bounded length and positive probability exists from every state. A common positive lower bound on that probability gives a geometric tail. The absorption probabilities are rational solutions of the finite linear system with boundary values one on the chosen outcome and zero on the other. The counts for the stated interval are given below. □

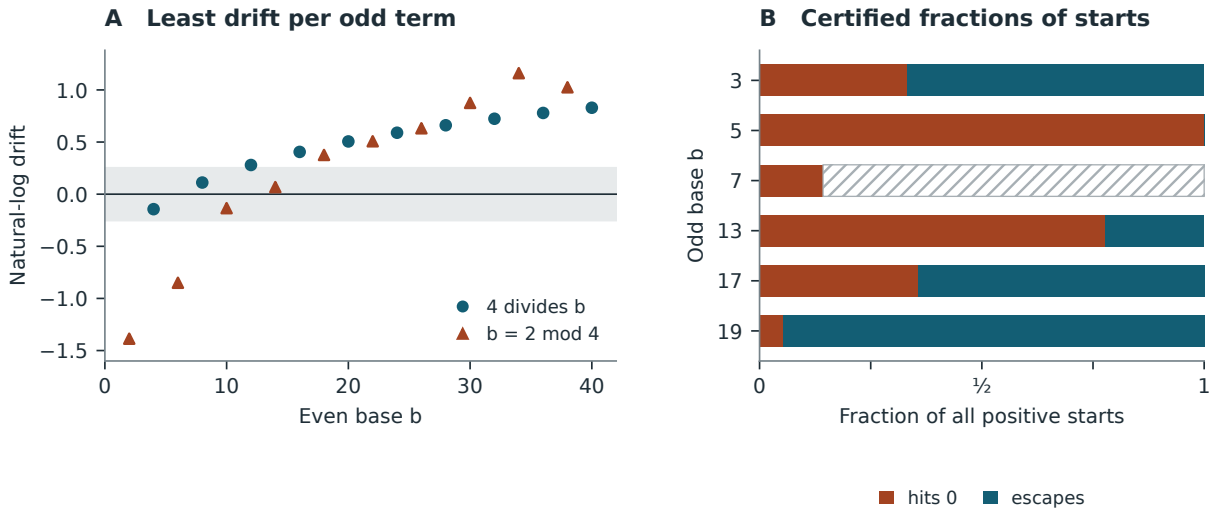


Figure 2: Left, each mark is a separate even base. The shaded band is $[-1/4, 1/4]$. Right, the orange part counts starts reaching a zero digit or multiple of the base; blue counts certified divergence. At base 7 the hatched fraction 6/7 has neither outcome certified by this graph. At base 17 the two measured fractions already sum to one, although a cycle through 9 exists.

b	τ_b	ρ_b	What else is known
3	1/3	2/3	$Z_b = 3\mathbb{N}$
5	1	0	Every start reaches 1
7	1/7	0	A period-20 orbit has least term 59
13	13210/16991	3781/16991	Complete absorption
17	148789/417911	269122/417911	A period-8 orbit has least term 9
19	1/19	18/19	A stretch enters an earlier escape class
25	1	0	Density-one trapping

Table 2: The fractions are absorption probabilities. Equality with the two natural densities follows when their sum is one. The base-7 trap density also follows directly from $Z_b = 7\mathbb{N}$.

Call a base *rooted* when -1 has a square root modulo b. Among the 999 odd bases from 3 through 1,999, all 235 rooted bases admit full absorption, with 27 trap-only and 208 split cases. Of 764 rootless bases, 670 are split and the graph leaves residual mass in 94. Stretches were tested through 2,000 steps, except in base 1,983, where a 6,000-step allowance resolved the remaining stretches (longest 2,640). Residual mass concerns the graph; its integer starts may exhibit additional behavior, as the explicit base-7 cycle demonstrates. A complete absorption statement, conversely, permits individual periodic exceptions of density zero.

4 Two refinements in base ten

4.1 Weighing paths backwards from one

The decimal residue law of the companion paper confines the odd basin of 1 to three classes modulo 15: $A = 1$, $B = 0$ and $C = 2$. If an odd predecessor m of an odd integer o reaches o after one multiplication and $v \geq 1$ halvings, then necessarily

$$m = (2^v o - 1)/d, \qquad d = m \bmod 10. \tag{4}$$

Substitution of the three classes gives the following exhaustive list.

Class of o	d	Class of m	Condition
A	1	A	$v \equiv 1 \bmod 4$
C	1	A	$v \equiv 0 \bmod 4$
A, C	5	B	$2^v o \equiv 1 \bmod 25$
B	7	C	$2^v o \equiv 1 \bmod 7$

No residue has two moves at the same v. Fix $Q = 3 \cdot 5^4 \cdot 7^3 = 643{,}125$, so 128,625 residues in A, B, C carry a weight. If o is known modulo Q, division by d in (4) determines m modulo Q/d. For an upper bound give this child the maximum weight of its d completions; for a lower bound use the minimum. Denote these weights by $w_{\max}(m)$ and $w_{\min}(m)$.

Lower bound. Set $\gamma = 399/800$. The integer vector in the supplement verifies at each admissible residue o that

$$\sum_{\text{moves at } o,\ v \le 64} (d/2^v)^\gamma w_{\min}(m) \ge w(o). \tag{5}$$

There are 2,038,400 included moves and $w(31)/\max w \ge .2017$. Write $N(o, X)$ for distinct descendants of o of size at most X, including o. For a basin member $o \ne 1$ and $X \ge o$, induction on $N(o, X)$ gives

$$N(o, X) \ge (2/5)\frac{w(o)}{\max w}(X/o)^\gamma.$$

Here is why truncating by size does not lose the induction. Retain children with $2^v o/d \le X$; each child is strictly smaller than that quantity. Their subtrees are disjoint, exclude o, and have smaller counts. The possible v's are spaced by $4, 20, 3$ for $d = 1, 5, 7$ respectively. For discarded moves, $\sum (Xd/(o2^v))^\gamma$ is less than $(1 - 2^{-8/5})^{-1} + (1 - 2^{-8})^{-1} < 5/2$ on A, C, or $(1 - 2^{-6/5})^{-1} < 5/2$ on B, since $\gamma \ge 2/5$. The cost after multiplication by $2/5$ is less than the one node o. Using (5) closes the induction. As $31 \mapsto 32 \mapsto 16 \mapsto 8 \mapsto 4 \mapsto 2 \mapsto 1$, the bound at $o = 31$ gives $.0806(X/31)^\gamma$. For $1 \le X < 31$, the term 1 suffices. Finally $31^\gamma < 5.544$ and $.0806/5.544 > .0145$.

Upper bound. For a reverse path let V be its total halvings and D its product of digits. A positive vector satisfying

$$\sum_{\text{moves at } o} d^s a^v w_{\max}(m) \le w(o), \qquad \min w \ge 1,$$

implies $\sum_{V=L} D^s \le w(o) a^{-L}$ by induction on L. Two vectors use $(a_H, s_H) = (.72135, 0)$ and $(a_P, s_P) = (.7116, 9/25)$. The first yields the concrete depth bound $H(L) \le 2953(.72135)^{-L}$ for the number $H(L)$ of odd nodes first reaching 1 at halving depth L. A three-state forward vector for $(a_F, s_F) = (.6331, 7/100)$ bounds the weighted coding of initial forward words. Its three rows sum the permitted geometric series on A, B, C; they are reproduced in the verifier.

For clarity, take $K = \lceil \log_2 X \rceil$, $\delta = .505$, and $\beta = .6245$. Follow the forward orbit while its cumulative halvings V stay strictly below K. If it reaches 1 in this interval, the depth bound gives $O(2^{\theta_H K})$ starts, where $\theta_i = \log_2(1/a_i)$. Otherwise the last prefix with $V < K$ has digit product D. If $D \ge 2^{\delta K}$, the forward vector and the companion coding injection give $O(2^{(\theta_F - s_F\delta)K})$ starts. Indeed the forward vector gives $\sum_{V=L} D^{s_F} \le (\sum h) a_F^{-L}$. A word of product at least $2^{\delta K}$ contributes at least $2^{s_F \delta K}$ to that sum. Sum the geometric bound over $L < K$.

In the remaining case the next step crosses K at a central odd node $o \le Y = 7\sqrt{K+1}\, 2^{\delta K}$. To see the square root, multiply the forward recurrence along the prefix including the crossing step. The distinct odd terms before the crossing are at least 15, and the product of $(1 + 1/(d_i n_i))$ is at most $\sqrt{K+1}$ by summing $1/(15 + 2i)$ over at most K distinct terms. If the last move is (d, v), the preceding path has $V < K$ and product D, then $o \le \sqrt{K+1} X d D / 2^{v+V}$. Raise this bound divided

by o to s_P, sum over preceding paths by the pair vector, and sum over moves. The sum over v is geometric. Since $2^{-s_P}/a_P > 1$, summing over $V < K$ leaves at most a constant times

$$(\sqrt{K+1}X)^{s_P}(2^{-s_P}/a_P)^K \sum_{\substack{o \le Y \\ o \text{ reaches } 1,\ o \text{ odd}}} o^{-s_P}.$$

If $N_1(t) \le Ct^\beta$ through Y, summation by parts bounds the last sum by $C\beta Y^{\beta-s_P}/(\beta - s_P)$. The resulting term is $O(C(K+1)^{\beta/2}2^{(\theta_P+\delta(\beta-s_P))K})$. The three exponents θ_H, $\theta_F - s_F\delta$ and $\theta_P + \delta(\beta - s_P)$ are all strictly less than β by integer comparisons. For large X, $Y < X - 1$ and the last term is at most $CX^\beta/2$. Choose C large enough for the first two terms and all smaller X to close induction on the integer size. This argument determines the upper exponent; the stated bound leaves C unspecified.

Depth is useful here for a structural reason. At a suitable odd multiple o of 15, the reverse moves $(7,1),(5,3)$ send o to $o' = (16o-15)/35$. Since $19o'+15 = 16(19o+15)/35$, choosing $5\cdot 35^r \mid 19o+15$ permits r successive double moves. Such odd o exist by solving $19(o/15)+1 \equiv 0 \bmod 35^r$. Each double move shrinks the result by a factor smaller than $16/35$. Consequently, on the full move graph on A, B, C, no finite residue vector bounds the descendant count at every positive starting node for *all* $X > 0$ by $w(o \bmod Q)(X/o)^c$ for $c > 0$: take $X < o$ equal to the last descendant and let r grow. These starting nodes are arbitrary odd multiples of 15 satisfying the stated congruence.

4.2 Which digit words can close?

Number the odd members $x_1, \ldots, x_m$ cyclically. Let d_i be the last digit and $v_i \ge 1$ the number of halvings in $2^{v_i}x_{i+1} = d_i x_i + 1$. With $K = \sum v_i$ and $D = \prod d_i = 3^a 5^e 7^c$, cancellation around the cycle gives

$$\frac{2^K}{D} = \prod_i (1 + 1/(d_i x_i)), \qquad 0 < \Lambda := K\log 2 - \log D < \frac{1}{X}\sum_i \frac{1}{d_i} \quad (x_i \ge X). \tag{6}$$

The next digit is determined by $d_{i+1} \equiv (d_i^2+1)2^{-v_i} \bmod 5$. In particular a 5 follows precisely a 3 or a 7, so their counts obey $n_5 = n_3 + n_7$. This turns the three prime exponents into four digit counts:

$$n_3 = e - c, \quad n_5 = e, \quad n_7 = c, \quad n_9 = (a - e + c)/2.$$

For the halving count set $w(1), w(3), w(5), w(7), w(9) = 1, 2, 0, 4, 3$, respectively, and $\phi(d) = \mathbf{1}_{d=5}$. The residues of 2^{-v} modulo 5 cycle in four steps, giving

$$v_i \ge w(d_{i+1}) + 4\mathbf{1}_{(d_i, d_{i+1})=(5,1)} + \phi(d_{i+1}) - \phi(d_i).$$

Indeed $3, 7$ can only enter 5 and require at least one halving. From $1, 9$ the least counts to $1, 3, 9, 7$ are $1, 2, 3, 4$; from 5 they are $4, 1, 2, 3$. The potential cancels around the cycle. If t_{51} counts $5 \to 1$ transitions then

$$K \ge n_1 + 2n_3 + 4n_7 + 3n_9 + 4t_{51}. \tag{7}$$

Short periods, with no size hypothesis. A length-one word reduces to $d = 1$ or 9 by the digit count identity. The identity $2^K = d + 1/x$ gives only $x = 1$. For two terms, other than the all-1 word, the digit multisets are $\{1,9\}, \{9,9\}, \{5,7\}$ and $\{3,5\}$. For $x_i \ge 3$, the first three give $9 < 2^K < 13$, $81 < 2^K < 88$, and $35 < 2^K < 40$. The last requires $K = 4$. Checking its three positive splits and the prescribed last digits yields only $(x_3, x_5) = (13, 5)$. If every digit is 1, the next odd member is at most $(x+1)/2 < x$ for $x > 1$.

For three terms, remove the known cycles. The possible least values for digits $1, 3, 5, 7, 9$ are now $11, 3, 15, 7, 9$. The digit-count identity leaves eight multisets. Of these one is all-1, and six enclose

no power of two when their digit minima are inserted into (6). Explicitly their intervals for 2^K are $(9,11),(81,91),(729,757),(135,154),(315,330),(35,40)$, in order for $119,199,999,359,579,157$ (here the three digits name a multiset). The last possibility, 135, forces $K=4$, yet its cyclic order is $3\to5\to1\to3$; (7) requires $K\ge7$. This proves the short classification through three odd terms by hand.

The same classification extends through *eight* odd terms by a finite computation with no bound on the initial integer. To bound the search itself, multiply the unscaled factors to obtain $2^K=\prod_i(d_i+1/x_i)\le10^m$. Let $L=\lfloor\log_2 10^m\rfloor$. Choose the first digit and positive $v_1,\dots,v_m$ with total at most L. There are exactly $5\binom{L}{m}$ choices, and the digit recurrence then determines the remaining digits. If after a prefix the current value is $(Ax+B)/2^h$, the next numerator has coefficients $(dA,dB+2^h)$. Closing the word forces $x=B/(2^K-A)$; a nonpositive denominator or a noninteger quotient is discarded. Test each surviving positive integer by its actual odd terms and exact halving counts. At $m=8$ this covers all 7,811,375 choices. At each odd length only start 1 survives; at each even length the starts are $1,5,13$. The two implementations in the supplement agree over the entire range. A primitive odd period is consequently either (1), (5, 13), or has at least nine odd terms.

Long cycles with a large minimum. Take $X=10^{15}$ and first suppose $K\le L=64{,}000$. Equation (6) gives $0<\Lambda<K/X$; in the weaker last bound one may use $m\le K$ and $d_i\ge1$. This puts (a,e,c,K) in a thin box. With $S=2^{58}$, put $\lambda_p=\lfloor S\log_2 p\rfloor$ and let $u_3=a,u_5=e,u_7=c$. The lattice coordinate $F=KS-a\lambda_3-e\lambda_5-c\lambda_7$ satisfies

$$1\le F\le\left\lceil\frac{3LS}{2X}\right\rceil+L,\qquad 0\le u_p\le\left\lfloor\frac{LS}{\lambda_p}\right\rfloor.$$

Indeed $F=S\Lambda/\log 2$ plus the three nonnegative floor errors. Their sum is at most $a+e+c\le K$, since $2^K>D\ge2^{a+e+c}$. The lattice is generated by $(0,0,0,S)$ and the three coordinate vectors with fourth entries $-\lambda_p$. The integer verifier reduces this basis, then enumerates every point inside the resulting bounds by exact rational arithmetic and decides (6) with interval logarithms. The test for a small *positive* difference is the part where ordinary floating-point logarithms can lose a candidate.

The gap leaves 298 candidates; 44 have nonnegative integral digit counts; 32 satisfy (6) with $n_1\le K-(n_3+n_5+n_7+n_9)$; and 19 survive the stronger (7). For each row (6) gives a lower bound on n_1 and hence on m. Their minima, taken separately, give the bounds in Theorem 3. In particular the row with $K=34{,}905$ requires $m\ge18{,}280$ and gives the total 53,185; the row with $m\ge15{,}496$ has $K=43{,}205$. A surviving row is a set of necessary conditions for a cycle. If $K>L$ and $9^m\le2^L$,

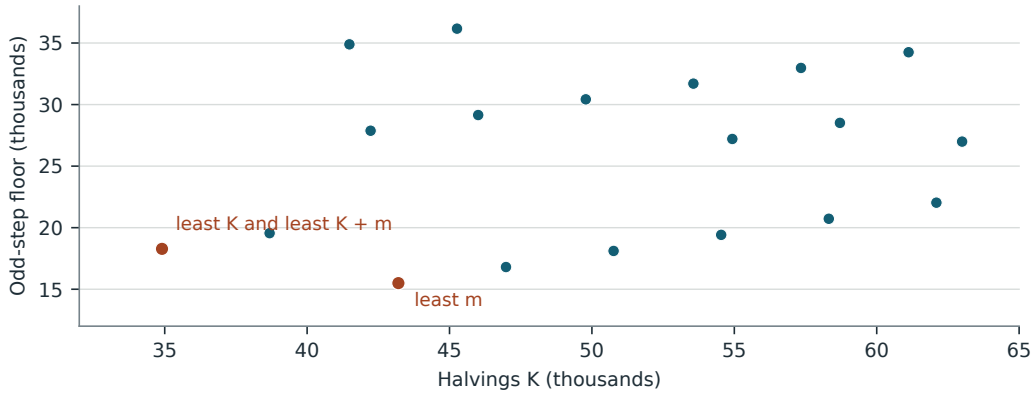


Figure 3: All nineteen surviving exponent triples represented by their halving count and odd-step floor. The two labeled points give three different minima. Adding the two coordinate minima would give too small a lower bound on the total period.

then $3m \le L$, hence $m/X \le 1/6$. But (6) implies $2^K < 9^m e^{m/X} < 2^{L+1}$, a contradiction. Direct integer powers give $9^{20189} < 2^{64000} < 9^{20190}$, so $m \ge 20{,}190$ in this tail. This proves the final part of Theorem 3. A separate exhaustive check in the companion paper covers every $7 \le n < 10^{15}$ descending below itself. A further cycle would have no such least member; the odd starts $1, 3, 5$ enter the two displayed cycles. Combining that record with the cutoff result therefore applies these three bounds to every further cycle.

5 Computational and formal verification

The reproducibility archive distinguishes a search for a witness from the check of that witness. The floating-point iteration that found the decimal weight vectors is dispensable once their integer entries are given. The checked lower vector has 104,125 positive entries. The two upper vectors contain 3,057,600 listed moves apiece through $v = 96$. The verifier rounds lower coefficients down and upper coefficients up with integer roots, adds a geometric bound $7^s(\max w)a^{97}/(1-a)$ to every upper row, and checks the integer inequality at every one of the 128,625 residues. It separately checks the three forward rows and exponent comparisons. The decimal growth bound uses these finite facts together with the analytic argument above.

For even bases the finite supplement builds the odd-term transition matrices from the formula for a digit. A second implementation solves stationarity in the shortcut chain for every even base through 208 and encloses each drift with rational logarithm intervals. It also checks finite-horizon probabilities a second way, by counting complete residue blocks of integers beyond all finite exceptions (78 base/horizon pairs). For odd bases it advances the known-bit cylinders, resolves containment and earlier escape classes, and solves the finite absorption systems. Neither calculation treats a sample of starts as a proof of a density. The cycle program re-enumerates the 19 rows using rational enclosures of the four logarithms; the candidate decisions and the published minima use integer or rational comparisons. The large decimal descent sweep is separate archived computational evidence, outside this download.

Formal scope. The `Ouroboros` Lean 4 library, pinned at `244ed2f9`, has 424 checked declarations across 30 source files; 174 are in the six modules devoted to this companion paper. Its axiom inventory contains only `propext`, `Classical.choice` and `Quot.sound`, or subsets thereof. The audit finds zero admitted declarations and zero uses of `native_decide`. The following are distinct proof steps.

Question	Lean kernel	Additional argument or computation
Even bases	Drift classification, finite arithmetic, parity coding, tilting inequalities	Finite-state ergodic theorem and last-minimum argument for all bases; explicit tilted weights
Odd bases	Cylinder rules, base-17 escape and density squeeze	The survey and its exact absorption solves
Decimal basin	Odd predecessor classification and the repeated contracting double move	Three large integer vectors, the forward rows and the induction on size
Decimal cycles	Halving budget, lattice-box inclusion, large-K tail and classification through three odd terms	Four-logarithm lattice enumeration, period check through eight, and the separate interval sweep

The README gives source files, theorem identifiers and the axiom receipt. In particular, the formal density squeeze takes its lower-density hypotheses as inputs. The individual base counts come from the cylinder calculation.

6 Prior work and open questions

Terras's stopping-time theorem for the classical map established the value of parity words for proving a natural-density result [2]. Matthews and Watts developed such coding for generalized Syracuse maps [3]. Leigh tracked residue information in a Markov process when nontrivial common factors obstruct simpler transition rules [4]. These are precedents for the coding used here. The explicit classification $\{2, 4, 6, 10\}$ and the zero-digit criterion concern the specific multiplier $n \bmod b$. Existing work on predecessor-tree inequalities and logarithmic restrictions on cycles supplies the closest methods for the decimal refinements [5, 6]. We have found no earlier classification of this particular family. The source ledger records the primary texts used and the companion manuscript.

Three questions survive the calculations. For base 8, can an actual positive integer be shown to tend to infinity, or must every orbit eventually recur? For base 25, does every positive start reach 1, including the possible exceptions left over by density-one trapping? For the decimal basin, does $\log N_1(X)/\log X$ have a limit, and does it equal the branching-model prediction $0.49944778\ldots$? The first concerns an infinite intersection; the second concerns possible density-zero exceptions; the third asks whether the residue distribution along an inverse tree can be controlled more sharply than its least and greatest completions.

7 Data, reproduction and authorship

The file `checker.zip` next to the manuscript runs directly with `python3 checker.zip`. It includes the integer vectors, both forms of the short-period check and the exact finite arithmetic. The figures have vector masters, print-resolution raster copies, source scripts and numerical receipts. The download includes the pinned Lean sources and axiom inventory. The original interval sweep stays in the research repository. The author used AI assistance to synthesize two earlier manuscripts, derive and test examples, prepare figures, and check the revised exposition. The manuscript awaits human mathematical review.

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