AI-assisted fourth draft for author review

Descent and competing basins in a decimal Collatz map

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Abstract

Halve a positive even integer. For an odd integer, multiply by its last decimal digit and add one. This map has cycles through one and five. We ask how often a starting value descends below itself and how many starting values reach each cycle. Every length-k string of shortcut parities occurs once for each starting residue modulo five. Counting these strings gives an exponential upper bound for the density of integers that have yet to descend. Looking backward, only three odd residue classes modulo thirty can reach one. A transfer matrix on these classes bounds their count by a power with exponent below 0.626383. For the cycle through five, a weighted tree rooted at three gives at least $0.02134X^{0.65}$ odd starts up to X, for every integer $X \geq 3$. The coefficient follows by charging a discarded branch its own weight in a finite certificate. Thus the ratio of the two basin counts tends to zero. An exhaustive computation finds that every positive integer through 10^{15} reaches one of the two cycles. Combined with an external integer lattice enumeration, it forces any further cycle to contain at least 53,185 ordinary steps. Universal attraction remains open.

MSC 2020 11B83, 11Y55, 37P99. **Keywords** Collatz maps, stopping times, inverse trees.

1 A digit chooses the multiplier

We call this the Ouroboros map. For a positive integer n, put $d(n) = n \bmod 10$ and let

$$f(n) = \begin{cases} n/2, & n \text{ even}, \\ d(n)n + 1, & n \text{ odd}. \end{cases}$$

The orbit of 31 goes to 32 and runs down the powers of two to 1. The orbit of 7 starts $7, 50, 25, 126, 63, 190$, climbs to 2576, and reaches 5 after 40 steps. Once inside either loop the path is prescribed:

$$1 \longrightarrow 2 \longrightarrow 1, \qquad 5 \longrightarrow 26 \longrightarrow 13 \longrightarrow 40 \longrightarrow 20 \longrightarrow 10 \longrightarrow 5.$$

Trace an orbit backward from its destination. Whenever two paths have the same final steps, they share a strand in Figure 1. The difference in the sizes of the two trees is visible before one counts anything. Sections below ask how much of that difference can be proved.

Santi recorded the latter loop in an earlier map that alternates the multipliers 3 and 5 on odd steps [1]. Its multiplier-5 turn at 5 and multiplier-3 turn at 13 reproduce the two odd moves here. His other two listed cycles leave 7 for 22 or 36. Here $f(7) = 50$.

The last-digit rule also makes sense in any integer base $b \geq 2$. At $b = 4$, write an odd integer as $2y+1$. Its next odd iterate is $y+1$ for even y and $3y+2$ for odd y; subtracting one and dividing by two returns the classical shortcut Collatz map on y. Yolcu, Aaronson and Heule give this change of variables for Farkas's variant [4]. The companion paper classifies the drift across even bases [5]. We take the decimal map on its own terms.

The following eight lines give the destination, time and peak of each opening example, counting until the first positive time at 1 or 5.

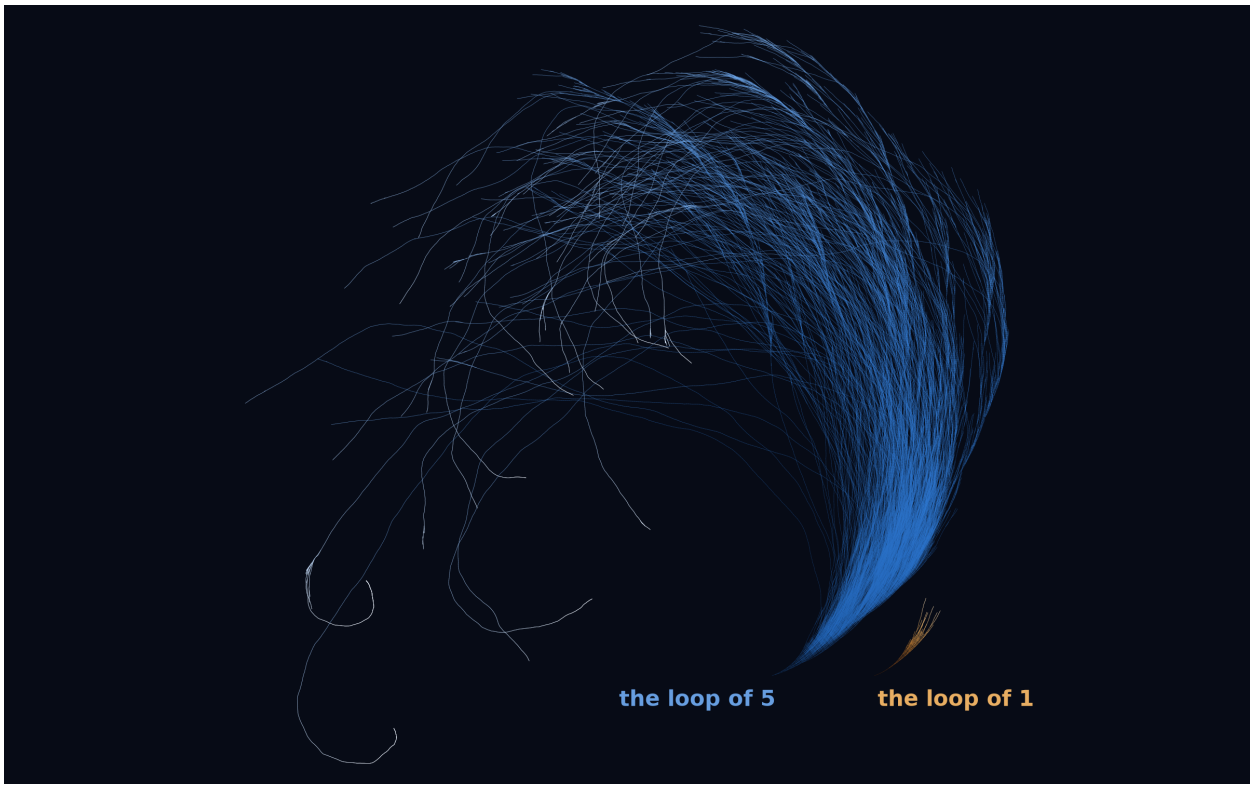


Figure 1: The two orbit corals. Each of the first 30,000 positive integers contributes its path to the first visit to 1 or 5. The blue coral has 29,813 starts reaching 5; the amber coral has 187 reaching 1. Read outward from the roots: a halving bends left and an odd move bends right. Shared segments represent identical iterates. Angles and lengths serve only to separate the strands.

```
def step(n):
    return n // 2 if n % 2 == 0 else (n % 10) * n + 1
for start in (1, 5, 7, 31):
    n, path = start, [start]
    while len(path) == 1 or n not in (1, 5):
        n = step(n)
        path.append(n)
    print(start, n, len(path) - 1, max(path))
```

The four output rows end in $(1, 2, 2)$, $(5, 6, 40)$, $(5, 40, 2576)$ and $(1, 6, 32)$, respectively. Figure 2 shows how little the size of the first step says about the ensuing excursion.

Three quantifiers lead to three different questions. Does a typical n have *some* smaller iterate? If so, how many starting integers can be traced to *each* known cycle? Can *every* positive integer be traced to one of them? Define $N_i(X)$ to count the odd positive $n \leq X$ whose orbits visit i, for $i = 1, 5$ and integral $X \geq 1$. We prove that the first question holds in natural density and that $N_1(X)/N_5(X) \to 0$. Neither theorem estimates the density of all starts reaching 5. For universal attraction, we have a finite verification through 10^{15} and restrictions on the least possible further cycle.

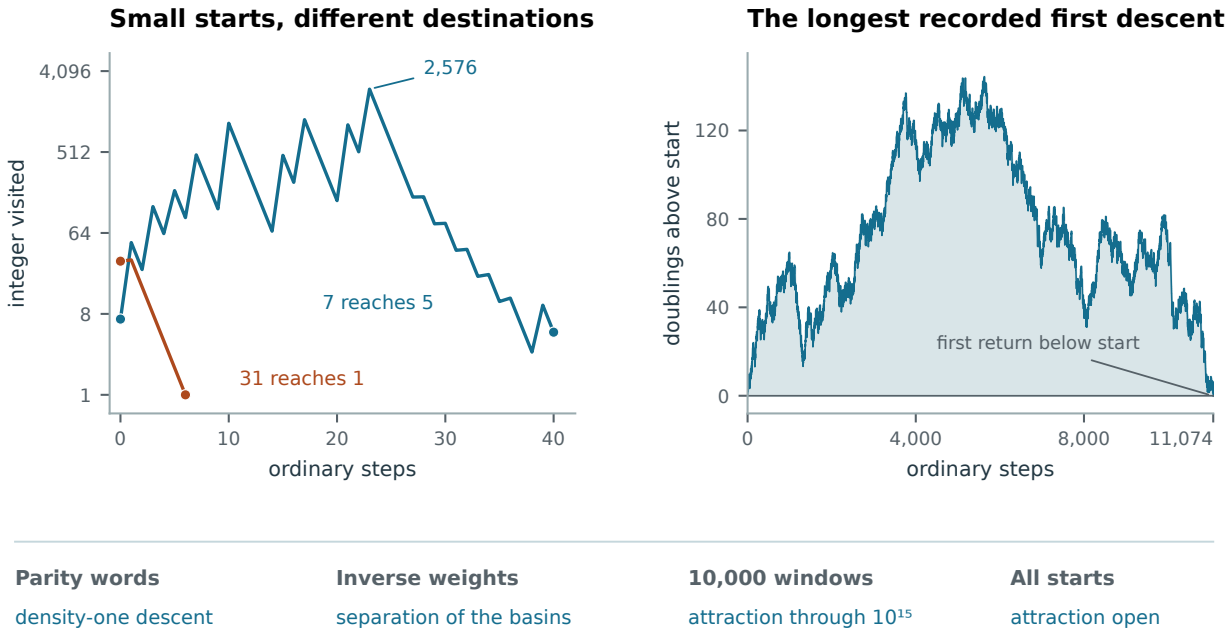


Figure 2: Ordinary iterates of 7 and 31 up to their first visit to 1 or 5, and of 889,676,852,592,557 up to its first strict descent. The second trace crosses its starting height after 11,074 steps. Its vertical coordinate is $\log_2(f^j(n)/n)$, so a height of one means twice the original integer. The decimal rule came from a search over 16 digit-based maps. The bottom row relates its three forms of evidence to the all-start question.

Why look at base ten?

The decimal rule emerged from a screen of 16 digit-driven maps. Its two observed cycles and the long climbs that preceded some returns suggested a comparison with the classical problem. Holding the digit rule fixed also makes it possible to vary the base alone.

Replacing the decimal last digit by $n \bmod b$ gives a controlled experiment: the starting integers stay fixed while the odd multiplier changes with the base. In Figure 3, both base four and base ten have long, irregular paths that come back down. All 60 sampled starts entered a cycle in each of bases four, six and ten. The median entry times were 279, 79.5 and 353.5 ordinary steps, respectively. Base eight has paths that climb through the plotted window. The companion paper gives the separate drift and stopping-density theorems for this family [5].

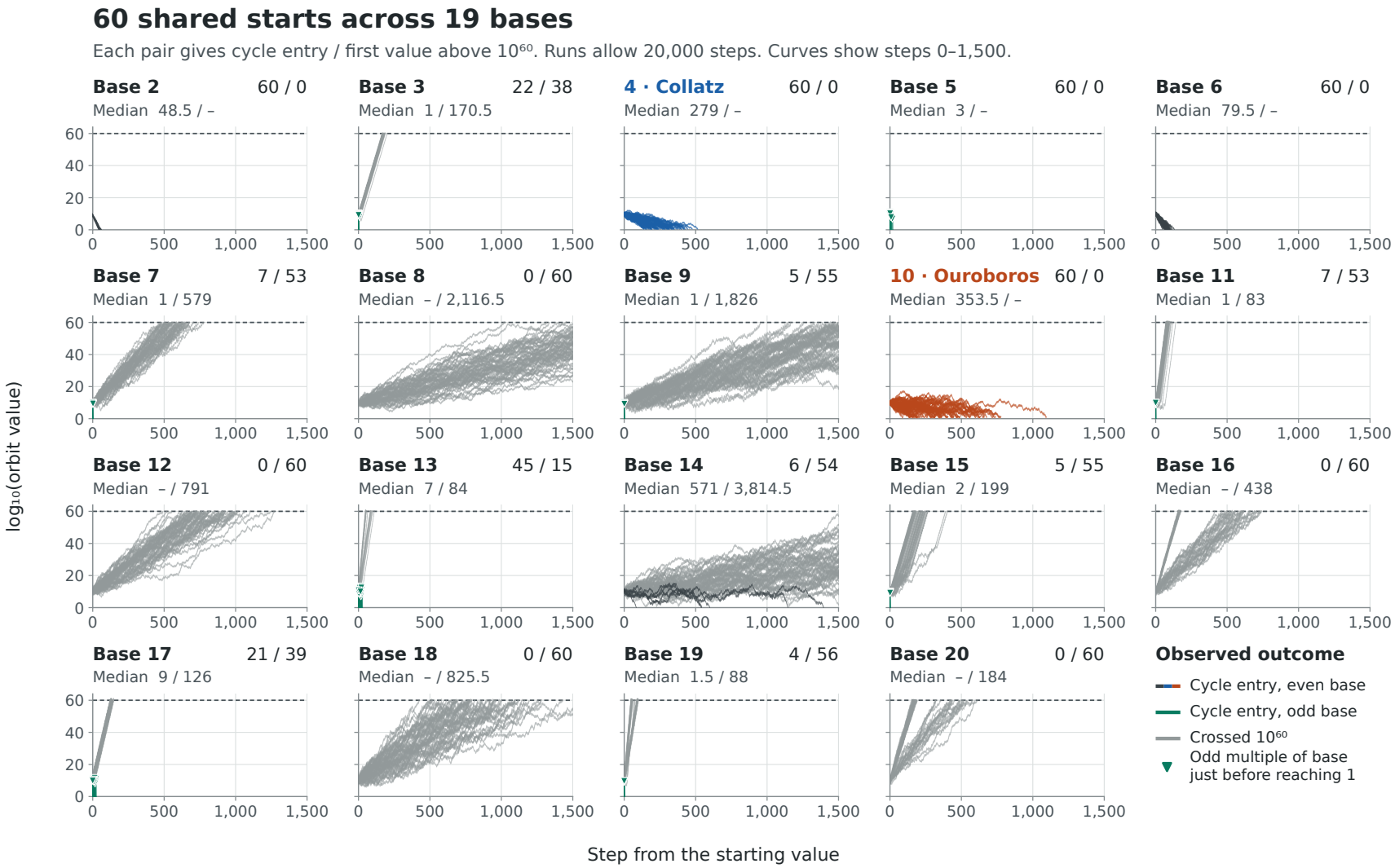


Figure 3: Each panel follows the same 60 seeded starts in $[10^9, 10^{10})$ in a different base from 2 through 20. Horizontal distance counts ordinary steps; vertical height is the decimal logarithm of the current value. Curves show up to 1,500 steps. The panel summaries use the full run to cycle detection or passage above 10^{60}, with a 20,000-step limit. In base 14 the six cycle entries are visible, while all 54 threshold crossings occur after the display ends. A crossing records a finite event.

A second comparison fixes the range of starting values and asks which paths share an ending. The two small central loops in the decimal panel of Figure 4 are the same loops that anchor the two corals. The starting samples differ; paths from the two samples can nevertheless share a final segment.

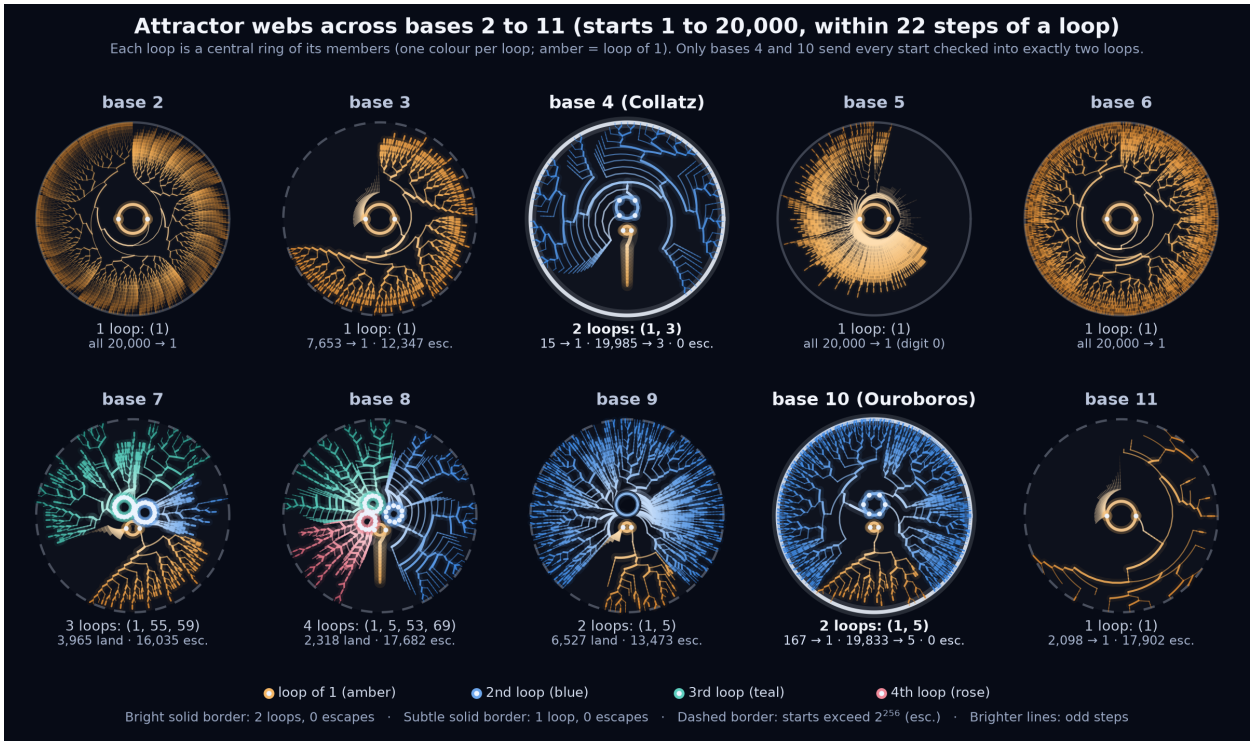


Figure 4: Paths near the observed cycles in bases 2 through 11. Each panel starts from all $1 \le n \le 20{,}000$ and draws up to 22 steps near a cycle. Central rings represent cycles, with individual members marked in small cycles. Outer rings count steps backward. Odd-base panels sample shallow spokes to keep the deeper branches visible. The escape counts printed in the panels count passages above 2^{256}. Neither experiment settles the fate of starts outside its range.

2 Forward in time: five states and one bit

Combine an odd move with its first halving. This gives the *shortcut* $T(n) = n/2$ for even n and $T(n) = (d(n)n+1)/2$ for odd n. Write the residue as $r = n \bmod 5$. An odd integer in class r has a unique last digit $\delta(r)$; in increasing order of r these are $(5, 1, 7, 3, 9)$. A halving multiplies a residue by 3 mod 5, so the possible next residues are

$$E(r) = 3r \bmod 5, \qquad O(r) = 3(r^2+1) \bmod 5.$$

In particular, a next *odd* term ends in 5 precisely when the previous odd term ended in 3 or 7. Additional halvings preserve divisibility by 5.

Lemma 1 (Counting parity words)**.** *Fix $k \ge 0$. A starting residue modulo 5 and a sequence of k shortcut parities specify precisely one class modulo $5 \cdot 2^k$. On every such class there are integers A_j (odd) and $B_j \ge 0$ such that $T^j(n) = (A_j n + B_j)/2^j$ for $0 \le j \le k$. The factor A_j is the product of the odd multipliers encountered.*

Proof. Start from $A_0 = 1, B_0 = 0$. An even step leaves both coefficients unchanged. An odd step replaces (A_j, B_j) by $(dA_j, dB_j + 2^j)$, with d fixed by the current residue. Advance that residue with E or O. Suppose the word through length j specifies n_0 modulo $5 \cdot 2^j$. The two lifts n_0 and $n_0 + 5 \cdot 2^j$ have jth iterates differing by $5A_j$, an odd number. Thus precisely one lift supplies each next parity. Induction gives both claims. □

One parity bit chooses the next residue

residue	digit	even →	odd →
0	5	0	3
1	1	3	1
2	7	1	0
3	3	4	0
4	9	2	1

Which parity words avoid a descent?

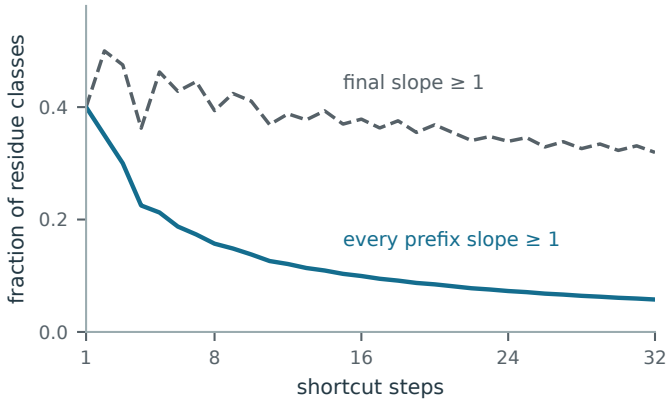


Figure 5: The two next residues and exact finite-word counts. For each length the denominator is $5 \cdot 2^k$. Counting a word after its last prefix tests only its final slope. Removing it as soon as a prefix contracts counts exactly the classes that avoid descent up to a finite set of starts.

For a uniformly chosen word and residue, each new parity is a fair bit. This is an exact counting statement for any fixed number of steps. For an individual integer the subsequent bits are determined by iteration. The one-step chain has stationary masses $(5, 3, 1, 4, 2)/15$ in residue order, as direct substitution in its two equally weighted columns shows. Conditioning on an odd step retains that distribution. The stationary logarithmic change from the multiplicative parts per odd step is

$$\sum_r \pi_r \log \delta(r) - 2 \log 2 = \tfrac{1}{15} \log(3^8 5^5 7 / 2^{30}) \simeq -0.134161.$$

The expected number of shortcut steps per odd step is two. The resulting drift describes the parity model. To prove an assertion on the integers we bound how many entire words can avoid contraction.

Reading from one odd term to the next gives another picture of the same mechanism. Every ribbon leaving digit 3 or 7 in Figure 6 ends at 5. The calculation above supplies the stationary weights; the theorem below turns counts of finite words into a statement about integers.

Theorem 2 (Exponential bound for first descent)**.** *Let E_k consist of $n \geq 2$ with $T^j(n) \geq n$ for $1 \leq j \leq k$. Then*

$$\overline{\mathrm{dens}}(E_k) \leq 1.124953\,(0.996423)^k.$$

The set of n having an ordinary f-iterate below n has natural density one.

Proof. For a word of length k set $H = A_k/2^k$. Whenever $H < 1$, the inequality $T^k(n) < n$ holds throughout its class for $n > B_k/(2^k - A_k)$. A fixed length has finitely many words. The upper density in question is at most the fraction with $H \geq 1$. In fact the density exists: for every

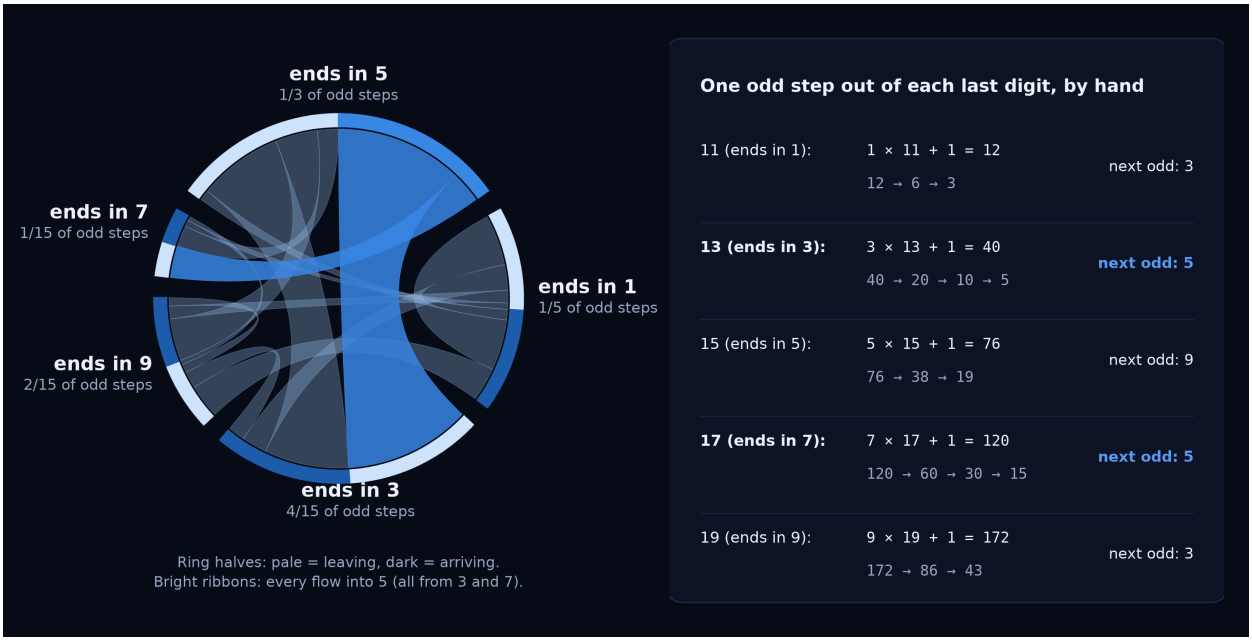


Figure 6: The five odd last digits, sampled at odd terms. Pale arcs emit stationary flow and dark arcs receive it; ribbon width measures the joint frequency of two successive digits in the model. The right panel works one odd-to-odd transition from each digit, including all intervening halvings.

nonempty prefix j its odd A_j differs from 2^j, so the sign of $T^j(n) - n$ is eventually constant within each class. This also proves the interpretation of Figure 5.

Take $t = 11/100$. On functions of the residue write

$$(Mu)_r = \tfrac{1}{2}(1/2)^t u_{E(r)} + \tfrac{1}{2}(\delta(r)/2)^t u_{O(r)}.$$

Then $\mathbb{E}[H^t] = \mathbf{1}^\top M^k \mathbf{1}/5$. Use the positive vector $u = (1000000, 837748, 965453, 964025, 944908)$. The bounds are

$$Mu < 0.996423\, u, \qquad \frac{\sum_r u_r}{5 \min_r u_r} = \frac{2356067}{2094370} < 1.124953.$$

Here is a rational check of the only fractional powers required. Put $D = 10^{12}$. In the order $d = 1, 3, 5, 7, 9$ take numerators

$$\begin{aligned} h_d = {} & 926588061891,\ 1045610747322,\ 1106046538591, \\ & 1147750484796,\ 1179922211263. \end{aligned}$$

Each satisfies $h_d^{100} 2^{11} > D^{100} d^{11}$. Direct substitution in the five rows gives

$$10^6 (h_1 u_{E(r)} + h_{\delta(r)} u_{O(r)}) < 2D \cdot 996423\, u_r.$$

Since M is nonnegative, $\mathbf{1} \le u / \min u$ and Markov's inequality give the claimed bound. A start with no smaller ordinary iterate belongs to every E_k. Letting k tend to infinity gives the density assertion. □

Terras's parity count for the classical map is the precedent [2]. Here the extra residue tracks the changing multiplier. The theorem permits a zero-density exceptional set; an orbit starting outside that set can still enter it after its first descent. Proving that every start reaches one of the two cycles would require control of later descents from those new starting values.

3 Backward toward one

Removing all powers of two after an odd move gives

$$F(n) = \frac{d(n)n + 1}{2^v}, \qquad v = \nu_2(d(n)n + 1), \quad n \text{ odd}.$$

Thus $F(991) = 31$, $F(31) = 1$. A different entry into that path is $F(195) = 61$ and $F(61) = 31$. The first example uses multiplier 1 at every odd move; the second uses 5 as well. The two constructions behind the basin bounds follow these examples.

An ordinary backward step and an odd-to-odd backward step measure different depths. Figure 7 takes ordinary steps, so the stem $1, 2, 4, 8, 16, 32$ appears as separate rings. The branch at 31 shows how a second odd number enters that stem.

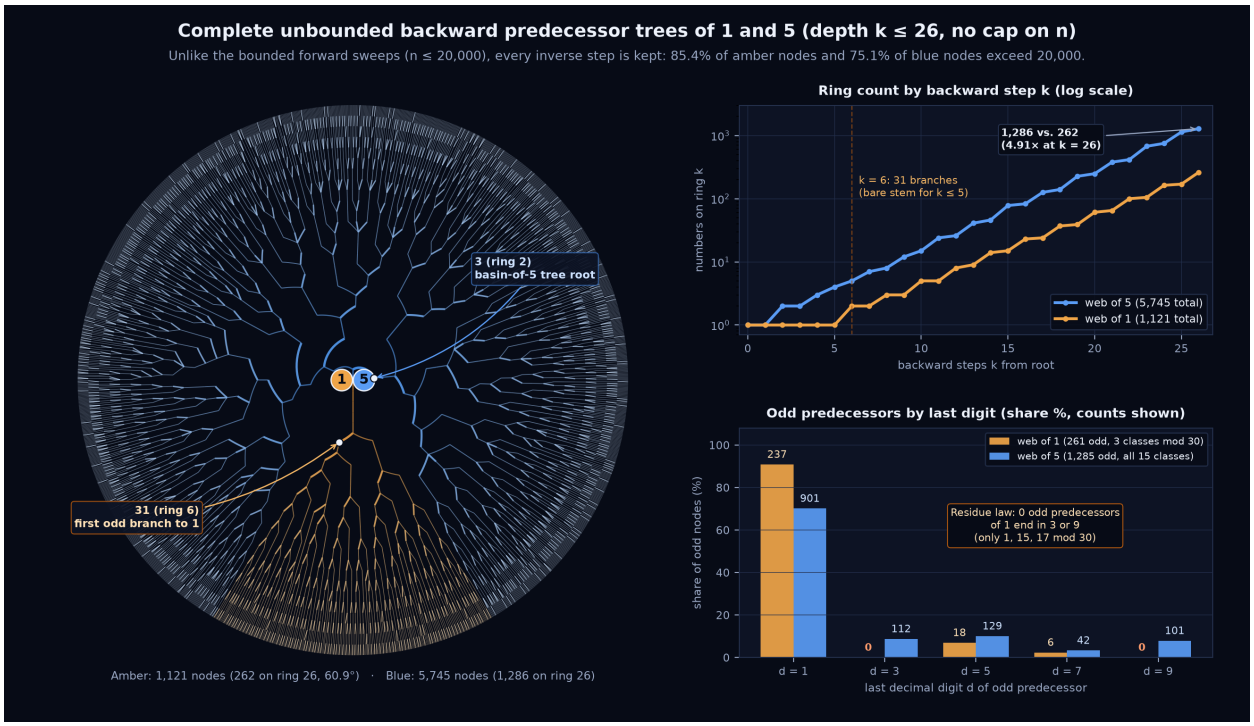


Figure 7: The distinct predecessors of 1 and 5 through 26 ordinary steps, with no ceiling on their values. Each node is assigned to its first-hit distance from the indicated root; periodic repeats are omitted. The outer ring has 262 nodes for 1 and 1,286 for 5. The lower-right panel counts the last digits of odd nodes, excluding the two roots. These ring counts fix a time horizon; the basin estimates below fix a bound on the starting integer.

Proposition 3 (Three residues for the basin of one)**.** *Every odd start reaching* 1 *is congruent to* $1, 15$ *or* 17 *modulo* 30*. Write* A, B, C *for these classes, or equivalently* $1, 0, 2$ *modulo* 15 *on odd integers. The possible transitions between them have the following halving counts.*

edge	$A \to A$	$A \to C$	$B \to A$	$B \to C$	$C \to B$
v	$1 \bmod 4$	$0 \bmod 4$	$0 \bmod 4$	$3 \bmod 4$	*all* $v \geq 1$

Proof. For a starting residue r modulo 15, the possible target residues are $(\delta(r \bmod 5)\, r+1)2^{-v} \bmod 15$. The multiplier 2 has order 4 modulo 15. For a target 1 the numerator must belong to $\{1, 2, 4, 8\}$;

the same set applies for target 2. For target 0 the numerator must vanish. In the order $r = 0, \dots, 14$ the numerators reduce to

$$(1, 2, 0, 10, 7,\ 11, 7, 5, 10, 7,\ 6, 12, 10, 10, 7).$$

Only $r = 0, 1, 2$ remain. Checking their powers of 2 gives the table. Backward induction from 1 proves the restriction. □

The paths containing only multiplier 1 have a binary description. Put $b = (n-1)/2$ for odd n. Such a path reaches 1 exactly when every run of ones in b has length divisible by four. Indeed those runs yield $15 \mid b$ by division into blocks of four ones, so n ends in 1. If b has r trailing ones, write $b = 2^{r+1}c + 2^r - 1$. Then $n + 1 = 2^{r+1}(2c + 1)$: the next test integer is c. The deletion shortens a positive binary string. Conversely, if both odd terms end in 1, reducing $n + 1 = 2^{r+1}(2c + 1)$ modulo five gives $2^r \equiv 1 \bmod 5$, so $4 \mid r$. Repeated deletions prove necessity. For example $(991 - 1)/2 = (111101111)_2$.

Theorem 4. *Let $a_1, a \in (0, 1)$ be the roots of $a_1 + a_1^4 = 1$ and $a + 2a^4 = 1$. Write $\gamma = -\log_2 a_1$ and $\beta = -\log_2 a$. For integral $X \geq 1$,*

$$a_1^9 X^\gamma \leq N_1(X) \leq 2.44 X^\beta, \qquad \gamma \simeq 0.464958, \quad \beta < 0.626383.$$

The basin of 1, including even starts, has natural density zero.

Proof. Build chains from 1 by the inverse steps $o \mapsto 2^v o - 1$, with $v \equiv 1 \bmod 4$ and first $v \geq 5$. These stay in A. Every child after the first is larger than its parent, and the valuation of $o + 1$ recovers the last step uniquely. The endpoint with total halving count $m > 0$ is strictly less than 2^m. Let $U(m)$ count ordered sums of m with positive parts 1 mod 4, including the empty sum. Its first five values are all 1. For $m \geq 5$, removing a first part 1 or subtracting 4 from a larger first part gives $U(m) = U(m-1) + U(m-4)$. Hence $U(m) \geq a_1^{4-m}$ by induction. Requiring the first part to be at least 5 gives $U(m-4)$ chains for $m \geq 5$. Taking $m = \lfloor \log_2 X \rfloor$ gives $N_1(X) \geq a_1^9 X^\gamma$ for $X \geq 32$; for smaller X the root alone suffices.

For the upper bound, expose successive valuations of F while their total is strictly smaller than an integer $k \geq 1$. A residue modulo 15 and the exposed tuple identify exactly one odd class modulo $15 \cdot 2^k$. To see this, fix the starting residue and digit d. The odd classes modulo 2^k map bijectively under $n \mapsto dn + 1$ to the even classes modulo 2^k. Its zero class gives the empty tuple. For a valuation $v < k$, division by 2^v leaves every odd class modulo 2^{k-v} once, with its residue modulo 15 fixed. Induction gives the assertion.

Weight a transition of valuation v by a^v. Summing all allowed valuations in each cell of Proposition 3 gives

$$P(a) = \begin{pmatrix} a/(1-a^4) & 0 & a^4/(1-a^4) \\ a^4/(1-a^4) & 0 & a^3/(1-a^4) \\ 0 & a/(1-a) & 0 \end{pmatrix}.$$

Why the second quartic? At its root set $h = (a/(1-a), 1, a/(1-a))^\top$. Substitution gives $P(a)h = h$. If $U_i(s)$ counts admissible tuples of total s from i, induction on s gives $U_i(s) \leq h_i a^{-s}$: the empty tuple costs at most h_i and adjoining a first transition uses precisely one term in $(Ph)_i$. Summing for $0 \leq s < k$ bounds the admissible classes by Ca^{-k}, with $C = (\sum_i h_i)a/(1-a)$. For $X > 15$ choose $k = \lceil \log_2(X/15) \rceil$. Each admissible class occurs at most once in $[1, X]$, and $Ca^{-k} \leq C(2/15)^\beta X^\beta < 2.44 X^\beta$. For example, bisection places a in $(0.6477988, 0.6477990)$; this interval suffices for the last constant and the exponent bound. For $X \leq 15$ the only possible additional odd candidate is 15, which maps through $76, 38, 19$; 19 is excluded by Proposition 3. Finally sum the upper bound over $X/2^j$ to include the even multiples of every odd start. □

4 Backward toward five: what a discarded branch costs

Krasikov and Lagarias used inverse-orbit difference inequalities to bound how many classical Collatz starts can reach a fixed target [3]. Here the root 3 leads to 10 and then 5 and never returns to 3. Its odd predecessors therefore form a tree. A child of a vertex o has the form

$$m = (2^v o - 1)/d, \qquad d \in \{1, 3, 5, 7, 9\},\ v \geq 1.$$

The single congruence $2^v o - 1 \equiv d^2 \bmod 5d$ guarantees integrality and the right last digit: the quotient is odd and congruent to d modulo five. It also makes $F(m) = o$ with exactly v halvings. For example $F(35) = 11$ since $5 \cdot 35 + 1 = 16 \cdot 11$. Thus even a branch that begins with a division by 5 can feed the same target as several digit-1 branches. We use only the finite set $1 \leq v \leq 16$.

Set $Q = 14175$ and use the five moduli $M_d = Q/d$. A type (M, r) says that the parent is congruent to r modulo M. For a full residue $R \bmod Q$ form the list $\mathcal{M}(R)$ of admissible pairs (d, v). Each $5d$ divides Q. Knowing $o \bmod Q$ therefore determines that list and the type of each child,

$$t'(R, d, v) = \big(Q/d,\ ((2^v R - 1)/d) \bmod (Q/d)\big).$$

There are 25335 types, the sum of the five moduli. Write w_t for their nonnegative integer weights and set $s = 13/20$. The certificate requires just two inequalities for each completed row:

$$\sum_{\mathcal{M}(R)} (d/2^v)^s\, w_{t'} \geq w_{(M,\, R \bmod M)} \quad \text{for every } M, \tag{1}$$

$$\sum_{\mathcal{M}(R)} w_{t'} \leq L. \tag{2}$$

The first inequality governs expansion through all available children. The second is a budget for the children that lie beyond the counting cutoff.

Lemma 5 (Charging individual child weights)**.** *Suppose the two inequalities hold with $L > 0$. For any odd vertex o in the tree of 3 and integral $X \geq o$, let $H(o, X)$ count the odd $n \leq X$ whose forward orbit visits o. For every type t of o,*

$$LH(o, X) \geq (X/o)^s w_t.$$

Proof. Put $q = X/o$ and $R = o \bmod Q$. Admit a pair (d, v) if $2^v o/d \leq X$, so its child m satisfies $m < 2^v o/d \leq X$. Distinct children have disjoint forward preimages because F is a function and o is nonperiodic. None of those preimages contains o. Hence $H(o, X) \geq 1 + \sum_{\text{admitted}} H(m, X)$ and each admitted $H(m, X)$ is strictly smaller than $H(o, X)$. Induction on this integer gives

$$LH(o, X) \geq L + q^s \sum_{\text{admitted}} (d/2^v)^s w_{t'}.$$

For a discarded pair $qd/2^v < 1$. The sum of all discarded terms, including the factor q^s, is at most $\sum_{\mathcal{M}(R)} w_{t'} \leq L$. The contribution of o covers that entire loss. Restoring the discarded terms and using (1) proves the bound, including the case of no admitted children. □

Theorem 6 (A weighted lower bound for the basin of five)**.** *For every integer $X \geq 3$,*

$$N_5(X) \ \geq\ \frac{98959}{2269527\, 3^{13/20}}\, X^{13/20} \ >\ 0.02134 X^{0.65}.$$

In particular $N_1(X)/N_5(X) = O(X^{-0.0236})$. The ratio also tends to zero when all positive starts, including even ones, are counted.

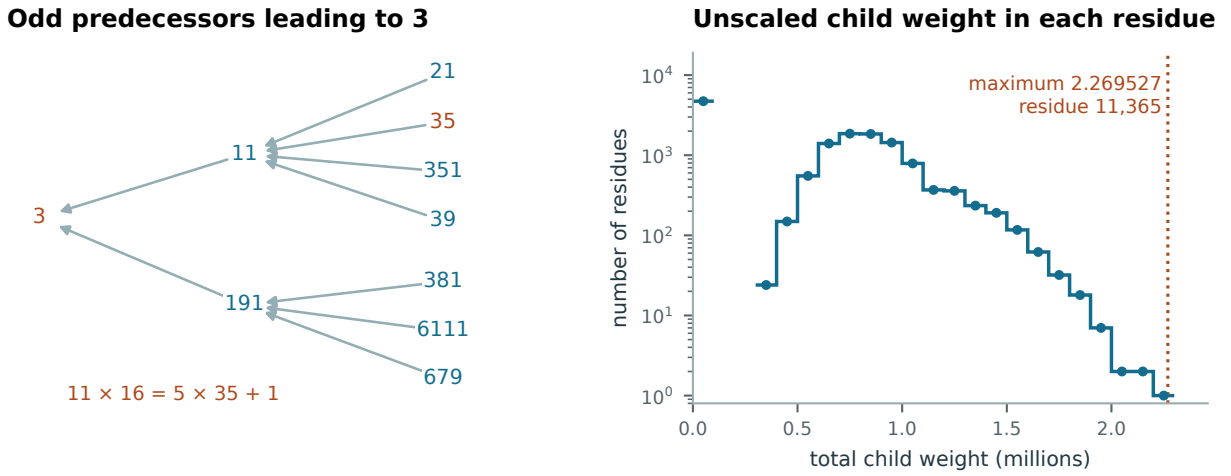


Figure 8: Left, odd predecessors at heights $v \le 6$ through two generations above 3; arrows point forward. Right, the sum of child weights for all 14,175 complete residues using $v \le 16$. The vertical line marks the largest amount one root could have to cover in (2). The logarithmic vertical axis counts residues in equal-width bins.

Proof. The supplied integer vector has $w_{(Q,3)} = 98959$ and largest entry 10^6. The finite verification can be described in full. For each d, v take the greatest integer $c_{d,v}$ for which

$$c_{d,v}^{20}\, 2^{13v} \le (10^6)^{20} d^{13}.$$

For each $0 \le R < Q$ initialize $S = W = 0$. For each $v = 1, \dots, 16$ and each $d \in \{1, 3, 5, 7, 9\}$, put $y = 2^v R - 1$. Whenever $5d \mid (y - d^2)$, add $c_{d,v} w_{t'}$ to S and $w_{t'}$ to W, where t' is the child type above. All 70,875 comparisons $S \ge 10^6 w_{(M, R \bmod M)}$ succeed, which proves (1). There are 81,072 admissible pairs over all rows, and their largest row sum is $W = 2{,}269{,}527$, attained at $R = 11{,}365$. This proves (2). Lemma 5 at 3 proves the first inequality. The rounding follows by raising to the twentieth power. Theorem 4 then gives the ratio because $0.65 - 0.626383 > 0.0236$. Its all-start numerator grows by at most a fixed factor after summing over powers of two, and its denominator includes all the odd starts. □

It is worth distinguishing the two uses of the finite table. Lean's kernel checks (1) for the same vector and the previous constant 0.000605. Its proof allows all 80 digit–valuation pairs to cost 10^6 each. The row sum L uses the actual child weights (Figure 8). The accompanying integer checker verifies this final calculation separately.

5 What the two cycles exclude

Suppose distinct odd vertices $x_0, \dots, x_{m-1}$ form a cycle under F. Set $d_i = d(x_i)$, $2^{v_i} x_{i+1} = d_i x_i + 1$, $K = \sum v_i$ and $D = \prod d_i$, with subscripts taken cyclically. Ordinary length is $K + m$. Multiplying around the cycle cancels the x_i:

$$2^K = \prod_i (d_i + 1/x_i), \qquad n_5 = n_3 + n_7, \tag{3}$$

where n_d is the multiplicity of digit d. The second equality counts arrivals at digit 5 using the succession rule of Section 2.

Proposition 7. *The only cycles with at most three odd members are the two displayed in Section 1.*

Proof. With one member the digit is 1 or 9. A sole 9 puts 2^K strictly between 9 and 10; a sole 1 forces $x = 1$. An all-1 cycle similarly contracts at each odd $x > 1$. Any other two-member cycle has digits $\{1, 9\}, \{9, 9\}, \{3, 5\}$ or $\{5, 7\}$. For $\{1, 9\}$ and $\{9, 9\}$ the minimum odd elements of those digits outside the 1 loop are 11 and 9, giving interval upper ends $(1 + 1/11)(9 + 1/9) < 10$ and $(9 + 1/9)^2 < 84$. The $\{5, 7\}$ interval lies between 35 and 38. None contains a power of two. For $\{3, 5\}$ the interval lies above 15 and below 18, so $K = 4$. A move $5 \to 3$ requires $v \equiv 1 \bmod 4$ and leaves three halvings for the reverse move. Solving $5x+1 = 2y$ and $3y+1 = 8x$ gives $x = 5, y = 13$.

A three-member cycle contains none of $1, 3, 5$: 3 reaches 5 and neither displayed period divides three except the fixed odd point 1 itself. The smallest remaining representatives of the five digits are $11, 13, 15, 7, 9$. The table gives the product D and the integer part U of the upper bound in (3) found by substituting these minima. The multiplicity constraint admits precisely the rows shown.

digits	111	119	135	157	199	359	579	999
D	1	9	15	35	81	135	315	729
U	1	10	17	39	90	142	329	756

Only the interval $(15, 17]$ contains a power of two. In that case the cyclic order must be $3 \to 5 \to 1 \to 3$. These edges require at least $1, 4, 2$ halvings respectively, by reducing $2^v d_{i+1} \equiv d_i^2 + 1$ modulo five. Their sum already exceeds $K = 4$. □

The companion also exhausts all ordered affine itineraries through eight odd members and obtains only repetitions of the two displayed cycles [5]. That extension uses finite arithmetic. The proof above through three odd members is formalized in Lean.

For long cycles the same cancellation turns a finite verification into a constraint on the period. The retained sweep tiles $[7, 10^{15})$ by 10,000 adjacent windows. Every odd start in that interval descends below itself. The search uses congruence classes for early descent and exact large-integer continuation when fixed-width arithmetic would overflow. The logs account for 499,999,999,999,997 odd starts, with 1,009,968,528 large-integer continuations. The endpoint 10^{15} is even and descends at once. The starts below 7 enter the two loops by inspection, so strong induction proves attraction throughout the verified range.

At the longest recorded first descent the first lower value is 821,329,135,081,529 (Figure 2). The largest reported peak needs 268 bits; two starts attaining it are 321,901,022,712,499 and 724,277,301,103,123. The accompanying checker replays both peak witnesses and the time record, and verifies the arithmetical accounting of every window. These checks have a different scope from rerunning the entire sweep.

Time and height along a finite orbit. The sweep records a first descent. Figure 9 follows each orbit farther, to 1 or 5. The thin amber band records how quickly the smaller basin returns in this sample; it is a different statistic from the time to fall below the starting value.

Different starts can attain exactly the same peak because they merge before the climb. The most populated band in Figure 10 collects 252,655 starts, the smallest of them 197. A single record orbit cannot show this multiplicity.

From a verified interval to a period. A further cycle would have all $x_i > X_0 = 10^{15}$. Taking logarithms in (3) gives the strict window

$$0 < K \log 2 - \log D < X_0^{-1} \sum_i 1/d_i, \qquad D = 3^a 5^b 7^c, \qquad (a, b, c) = (n_3 + 2n_9, n_5, n_7).$$

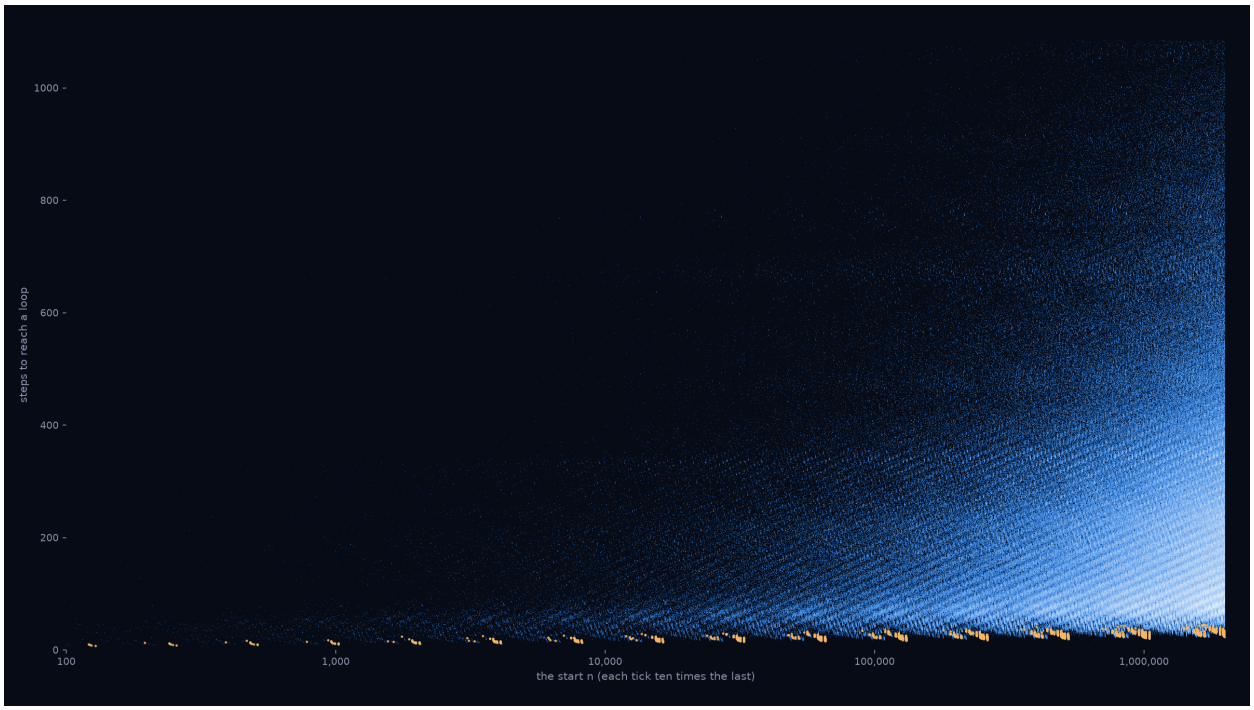


Figure 9: First-hitting times for the two roots, with the starting integer on a logarithmic horizontal axis. The computation covers $1 \le n \le 2{,}000{,}000$; the plot begins at 100 and its vertical axis ends at 1,085 steps. In the full census 7,762 starts took longer, including the path of 3,337 steps from 1,761,259. Blue density counts paths to 5; amber dots end at 1.

For $K \le 64000$ one has $m \le K$ and the gap is smaller than $\log 2$, so $2^{K-1} < D < 2^K$. The companion's integer lattice search exhausts the exponent triples in this box [5]. To see what a survivor means, $n_3 = b - c$ and $n_9 = (a - b + c)/2$ must be nonnegative integers. The halving counts restrict the remaining free multiplicity n_1 by

$$2n_1 + 3a + b + 7c \le 2K.$$

Indeed give incoming digits $1, 3, 5, 7, 9$ the weights $1, 2, 0, 4, 3$ and put $\phi(d) = \mathbf{1}_{\{d=5\}}$. Reducing modulo five gives $w(d') + \phi(d') \le v + \phi(d)$ for every admissible $d \to d'$. For $d = 1, 9$ the minimum v to reach $1, 3, 7, 9$ is $1, 2, 4, 3$. For $d = 3, 7$ the target is 5 and $v \ge 1$ suffices. For $d = 5$ the minimum v to reach $3, 9, 7, 1$ is $1, 2, 3, 4$. Summation around the cycle cancels ϕ and gives the displayed restriction. For each triple the strict logarithmic window also gives a lower bound on n_1.

The external enumeration returns 298 candidates for the first logarithmic window with $\sum_i 1/d_i \le K$, 44 satisfying the exact digit constraints, 32 satisfying the basic $m \le K$ budget with the weighted window, and 19 surviving the displayed restriction. The three separate minima over those survivors are

$$K \ge 34905, \qquad m \ge 15496, \qquad K + m \ge 53185.$$

For instance $(a, b, c) = (8842, 7001, 1651)$ gives $K = 34905$ and $m \ge 18280$, while $(13041, 4567, 4250)$ gives $K = 43205$ and $m \ge 15496$. These are possible exponent vectors; a cycle would also have to realize their entire ordered itinerary. For a direct check of the remaining tail, if $K > 64000$ and $9^m \le 2^{64000}$, then $D \le 2^{64000}$ and $m \le 64000$, so the gap would be at least $\log 2$ and less than $64000/X_0$, a contradiction. As $9^{20189} < 2^{64000}$, this tail has $m \ge 20190$ and length at least 84191. Thus the displayed minima hold for the full range.

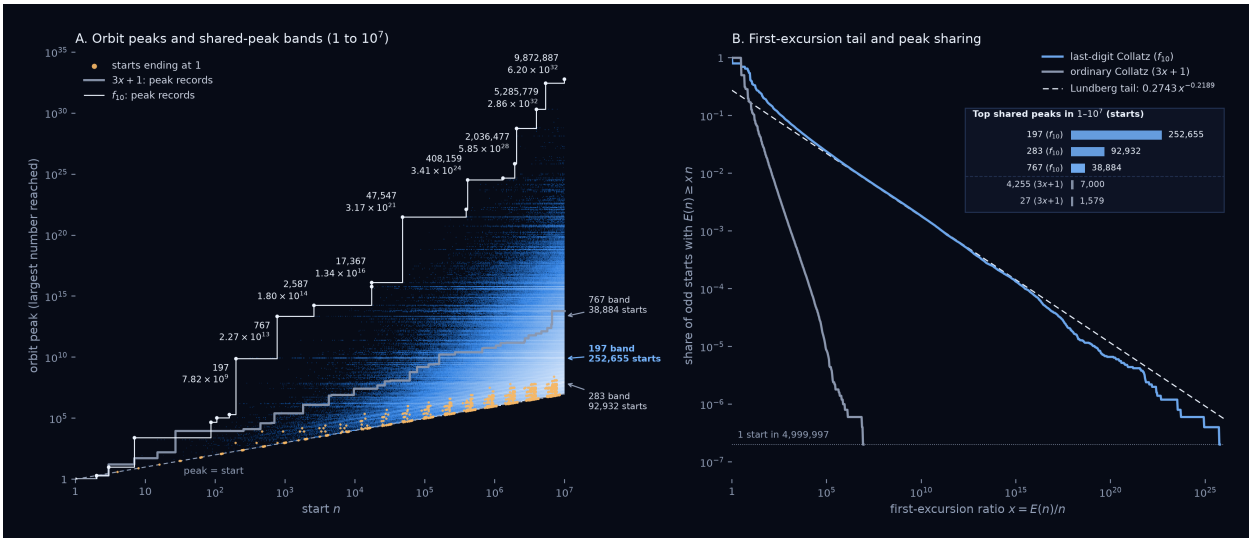


Figure 10: Left: peak height against starting value through 10^7, with record heights for the last-digit and classical Collatz maps. Horizontal bands collect starts with the same maximum up to the first visit to 1 or 5 (to 1 for the classical map). Right: the fraction of odd starts whose peak before first descent reaches a given multiple of their start ($7 \le n \le 10^7$ for the decimal map, $3 \le n \le 10^7$ for the classical map). The dashed power law is a reference curve. The inset shows selected peaks, including the most shared in each map and the peak reached by the classical path from 27.

6 How much of the basin law is known?

For $X = 10^3, 10^4, 10^5, 10^6$, direct iteration gives $N_1(X) = 11, 34, 101, 349$. Among the first 300,000 positive integers of both parities, 668 reach 1 and 299,332 reach 5. For upper cutoffs beyond the verified range neither the density theorem nor the basin ratio counts the orbits that evade both known cycles.

The inverse tree with every odd iterate capped at 2^{72} contains 383,455, 12,091,490 and 381,309,367 odd starts at cutoffs 2^{40}, 2^{50} and 2^{60}. Each count is a lower bound on N_1: an orbit can pass above the ceiling on its way to 1. In this capped tree the count increases about thirty-twofold per ten doublings of the cutoff. The companion's larger certificates improve the exponent bounds to $0.0145X^{0.49875} \le N_1(X) \ll X^{0.6245}$ [5]. That interval leaves open whether $N_1(X) = X^{\alpha+o(1)}$ for any single exponent α. A model that averages branching over congruences modulo 25 and 7 predicts $\alpha \simeq 0.4994477848$. Its characteristic equation reduces, with $a = 2^{-\alpha}$, to

$$1 = a + (1 + 35^{\alpha-1})a^4.$$

Indeed the original determinant has the additional positive factors $(1-a)(1-a^4)$. The interior root is unique. Writing the right-hand side as $2^{-\alpha} + 2^{-4\alpha} + (35/16)^{\alpha}/35$, its first term alone has negative derivative of magnitude at least $(\log 2)/2$ on $[0,1]$, while the positive derivative is at most $\log(35/16)/16$. The right-hand side crosses one between the endpoints. A proof of the predicted exponent would need suitable control of residues in the actual tree. An independent open question is whether any start avoids both cycles forever.

Figure 11 compares the capped-tree counts with the direct census where both are available. For odd starts below 2^{60}, raising the permitted odd iterate from 2^{60} to 2^{72} recovers 10,355,985 additional starts that reach 1. This is how a ceiling can affect the count even below the starting cutoff. Normalizing the counts makes the variation within each doubling visible even where the logarithmic growth plot looks nearly straight.

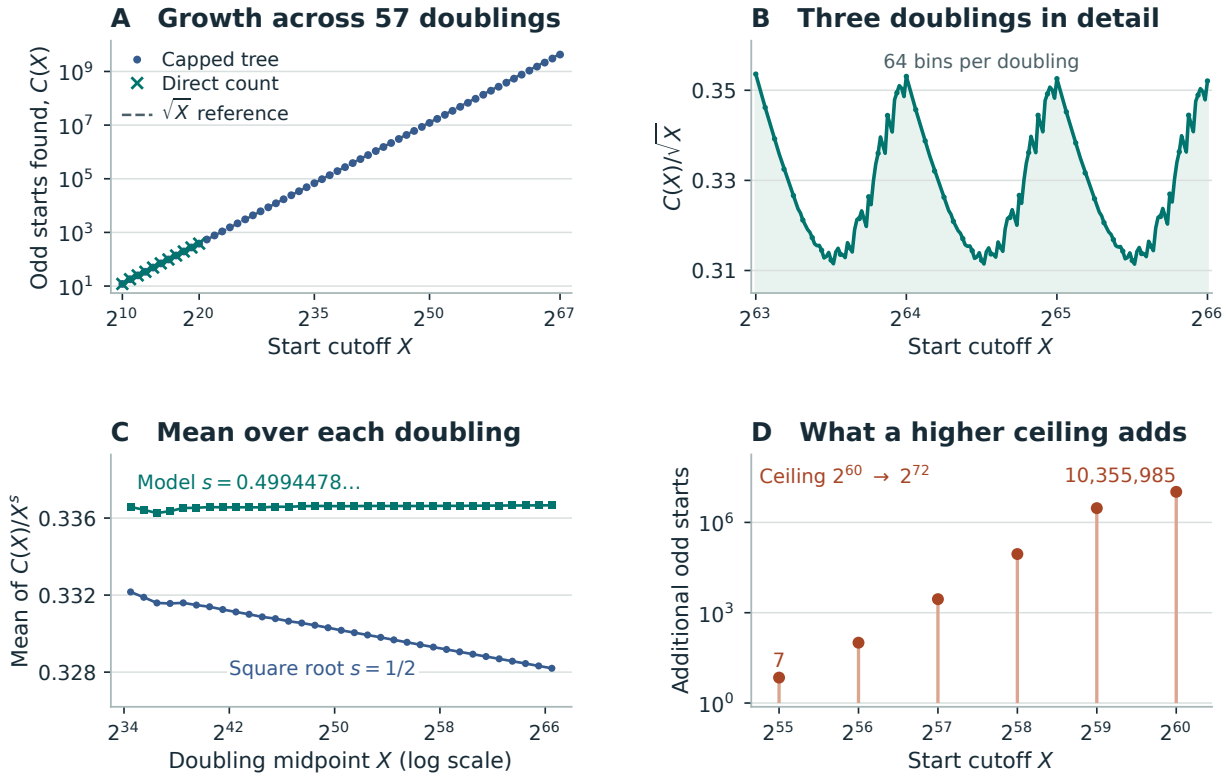


Figure 11: Odd starts below X reaching 1 with every odd iterate at most 2^{72}. (A) Direct forward iteration gives the same counts through 2^{20}. At larger cutoffs the plotted counts are lower bounds for the full basin. (B) Division by $\sqrt{X}$ reveals variation within each doubling. (C) Geometric means over 64 bins compare square-root growth with the model exponent $\alpha \simeq 0.4994478$. (D) Additional starts recovered by raising the odd-iterate ceiling from 2^{60} to 2^{72}.

Data, formalization and authorship. The downloadable checker executes the printed example, verifies the five-state matrix and every inverse-tree inequality, and recovers the period minima from the supplied digit-valid triples. It includes the integer weights and all window summaries. An optional run follows both record trajectories in every window. The original research archive retains the sweep code, raw logs and lattice enumerators. The archived Lean 4 development covers parity coding and first descent, the quartic bounds, the older certificate bound for five, the short-cycle argument and the algebraic reduction of long cycles. At revision `d0d72b27` the audited inventory contains 424 declarations across the decimal and companion developments. Its axiom report uses only propositional extensionality, classical choice and soundness of quotients, with no proof placeholders. The weighted terminal sum in Theorem 6, the large sweep, the finite lattice enumeration and the refined companion exponents use the stated integer computations outside that kernel. The first author designed and directed the AI-assisted search that found this decimal rule among 16 digit-based candidates. AI agents developed the proofs, computations and exposition under his direction.

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