AI-assisted third draft for author review

The most unit cubes a plane can cut

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Abstract

Subdivide a cube into n^3 unit cubes and pass one plane through it. A cut counts whenever the plane enters the interior of a unit cube, however small the piece it removes. Bárány and Frankl found a slightly tilted plane that cuts $\lfloor (9n^2 + 4n - 17)/4 \rfloor$ cubes. We prove that their construction is optimal for every $n \geq 3$. In particular, a ten-by-ten-by-ten block admits exactly 230 cuts. The upper bound counts diagonal chains of cubes that a plane must miss. The starts and ends of these chains form six lattice triangles. Pairing rows in opposite triangles eliminates the height of the plane. For even n, putting the height back into just one pair recovers a single missing unit. The resulting inequalities have two slope variables. A uniform estimate proves them for $n \geq 700$, and finite certificates cover the smaller sizes. Lean's kernel checks the small range through 32 and the large-size argument. The intervening 667 sizes use two independent exact-integer evaluations of a kernel-proved sufficiency criterion.

MSC 2020 52C07, 52C35, 11H06. **Keywords** lattice cubes, plane sections, digital planes.

1 A cut worth seven more cubes

In a block ten cubes on a side, the best plane perpendicular to a body diagonal enters 223 cubes. Tilt it by just enough to pick up seven more, and every one of the original 223 survives. Could a second tilt, or a different height, pick up an eighth? Bárány and Frankl bounded $m(10)$ between 230 and 246. Separately, their author manuscript asserted a further gain of two cells for all sufficiently large n [2, §5].

Theorem 1. *Let $m(n)$ be the largest number of open unit cubes of $[0, n]^3$ that one plane can meet. Then $m(1) = 1$, $m(2) = 7$, and*

$$m(n) = F(n) := \left\lfloor \frac{9n^2 + 4n - 17}{4} \right\rfloor \qquad (n \geq 3). \tag{1}$$

The lower bound belongs to Bárány and Frankl. What needs proof is optimality. Equation (1) also rules out the proposed further gain. At $n = 10$ an integral normal $(9, 9, 10)$ and a half-integral height give a witness whose strict inequalities can be checked in seven lines.

```
from itertools import product
n = 10
h = n-1
s = (3*n-4)//2
twice_height = 2*h*(s+3)+1
lows = (h*(i+j+k)+k for i,j,k in product(range(n), repeat=3))
assert sum(2*q < twice_height < 2*(q+3*h+1) for q in lows) == 230
```

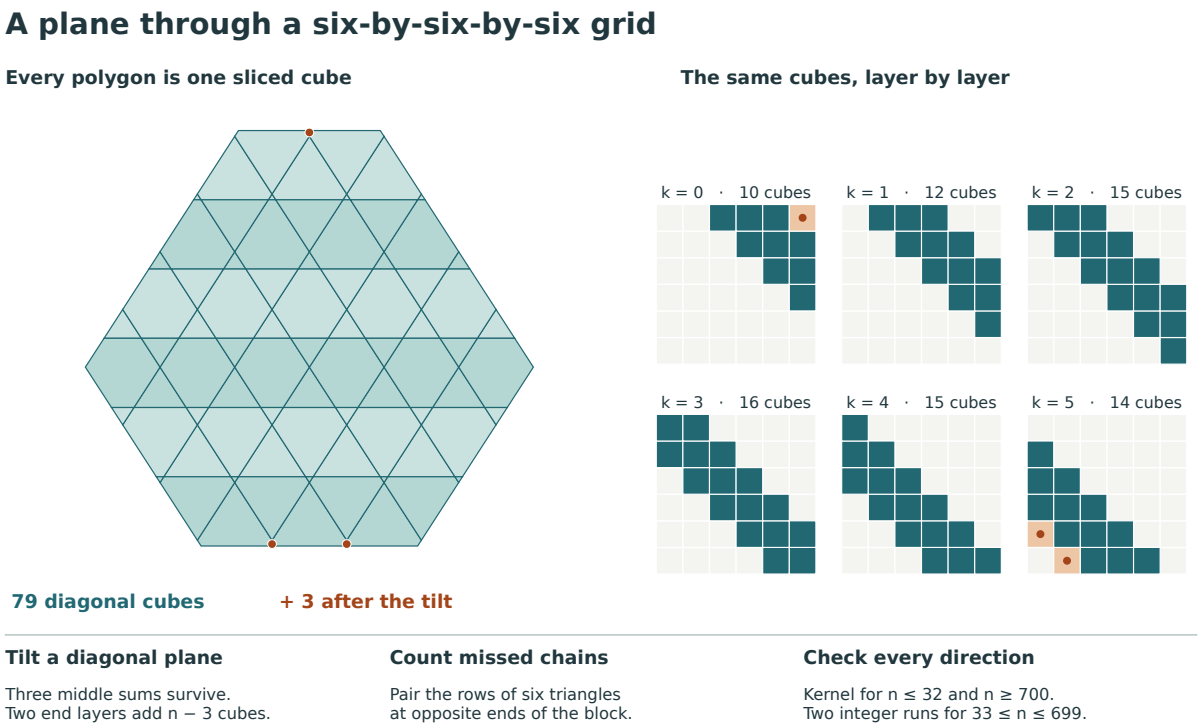


Figure 1: An extremizing section of $[0,6]^3$, seen face-on at left and from above in each horizontal layer at right. The plane $5x + 5y + 6z = 101/2$ retains the 79 cubes of the shifted diagonal slice $x + y + z = 19/2$. Dots mark the three extra cubes. In every map i increases rightwards and j upwards. The lower strip gives the proof coverage of Theorem 1.

The witness gains exactly $n - 3$ cubes over the best translate of the diagonal plane. For the upper bound it is easier to count absences. Following Bárány and Frankl [2, §6], line up the cubes in chains running parallel to $(1, 1, 1)$. Each plane meets at most one cube per chain after reflection makes its normal nonnegative. Some chains start after the plane has passed, others end before it arrives. A pair of opposite corners captures both effects. Increasing the height subtracts exactly as much from one corner threshold as it adds to the other. Counting antidiagonals makes that cancellation survive integer rounding.

2 A witness and a convention about contact

Write $h = n - 1$. The cell with lower corner $z = (i, j, k) \in \{0, \dots, h\}^3$ is $z + (0, 1)^3$. Reflecting coordinate axes and translating the height lets us take the normal $v = (a, b, c)$ nonnegative. Write $W = a + b + c > 0$. The image of the open cell under $v \cdot x$ is the open interval $(v \cdot z, v \cdot z + W)$. In particular, a plane that merely touches an edge contributes nothing to that cell. Moving a plane sufficiently little preserves every cell it already cuts.

Proposition 2. *For $n \geq 3$, put $s = \lfloor (3n - 4)/2 \rfloor$. The plane*

$$hx + hy + nz = h(s + 3) + \tfrac{1}{2} \tag{2}$$

cuts $F(n)$ cells. Exactly $n - 3$ are additional to a best diagonal translate.

Proof. Set $\sigma = i+j+k$. The integral endpoints of the cell interval are $h\sigma+k$ and $h(\sigma+3)+k+1$. Inserting the half-integer height shows that every $\sigma = s, s+1, s+2$ occurs. Two further pieces survive, $\sigma = s-1$ with $k = h$, and $\sigma = s+3$ with $k = 0$. These are the edge strips visible in Figure 1.

Let $H(r)$ be the coefficient of z^r in $(1+z+\cdots+z^h)^3$. The best diagonal translate collects the three most central consecutive coefficients, $H(s)+H(s+1)+H(s+2)$. Indeed, convolution with the indicator of $\{0,\dots,h\}$ preserves symmetry and unimodality. Consecutive window sums differ by the arriving coefficient minus the departing one. Start with the constant coefficient sequence of the first factor and apply this argument twice. A diagonal plane through a corner can only lose endpoint layers.

Set $\binom{r}{2}_+ = r(r-1)/2$ for $r \geq 2$ and zero otherwise. Expanding $(1-z^n)^3/(1-z)^3$ gives $H(r) = \sum_{j=0}^{3}(-1)^j\binom{3}{j}\binom{r-jn+2}{2}_+$. There are $\max(0, n-|r-h|)$ pairs in $\{0,\dots,h\}^2$ of sum r. Apply this last formula to $r = s-1-h$ and $s+3$. Substitution gives

	$H(s)+H(s+1)+H(s+2)$	two strips	total
$n = 2q$	$9q^2-2$	$2q-3$	$9q^2+2q-5$
$n = 2q+1$	$9q^2+9q+1$	$2q-2$	$9q^2+11q-1.$

The last column is (1) in both parities. □

3 Which chains must a plane miss?

Normalize $W = 1$ and order the normal $x_1 \leq x_2 \leq x_3$. A diagonal chain starts at an index with minimum coordinate zero, advances by $(1,1,1)$ and ends when its maximum coordinate is h. There are

$$C = n^3-(n-1)^3 = 3n^2-3n+1, \qquad D = C-F(n). \tag{3}$$

Successive lower endpoints on a chain are one unit apart. An open interval of length one contains at most one of them. We therefore need D distinct missed chains.

This already settles $n \leq 4$. At $n = 1$ a plane through the center suffices. At $n = 2$ the chain bound is 7, attained by $x+y+z = 5/2$. At $n = 3$ the chain bound 19 equals $F(3)$. At $n = 4$ use the four one-cell chains with indices $(0,3,0),(0,3,3),(3,0,0),(3,0,3)$. Within each displayed pair the endpoint difference is $3x_3 \geq 1$, so at least two of the 37 chains are missed. Proposition 2 supplies the matching 35.

For $n \geq 5$ two reductions remove unhelpful orientations and heights. If $x_3 \geq 1/2$, every vertical column contributes at most two cells, hence at most $2n^2 \leq F(n)$. Otherwise $x_1 \geq 1-2x_3 > 0$. A small perturbation preserving the finite set of strict cell inequalities lets us assume $0 < x_1 < x_2 < x_3$ and $x_2-x_1 \neq x_3-x_2$. Reflect the whole cube to take $t \leq n/2$. While $t < hx_3$, no cell with $k = h$ is cut. Sending k to $k+1$ and t to $t+x_3$ preserves all cuts. The first resulting $t \geq hx_3$ is less than $nx_3 \leq n-hx_3$. We may thus work with

$$hx_3 \leq t \leq n-hx_3. \tag{4}$$

For each k, call the other two weights $p_k \leq q_k$, and set $s_k = p_k+q_k = 1-x_k$, $d_k = q_k-p_k$. Write

$$\Delta(p,q;b) = \#\{(i,j) \in \mathbb{N}_0^2 : pi+qj < b\}, \quad \alpha_k = hs_k-t, \quad \beta_k = t-hx_k-1.$$

Consider a chain starting on $z_k = 0$. If its lower endpoint already exceeds t, it is missed. Measured inward from the far corner of that face, its indices satisfy $p_ki+q_kj < \alpha_k$. The triangles fit inside

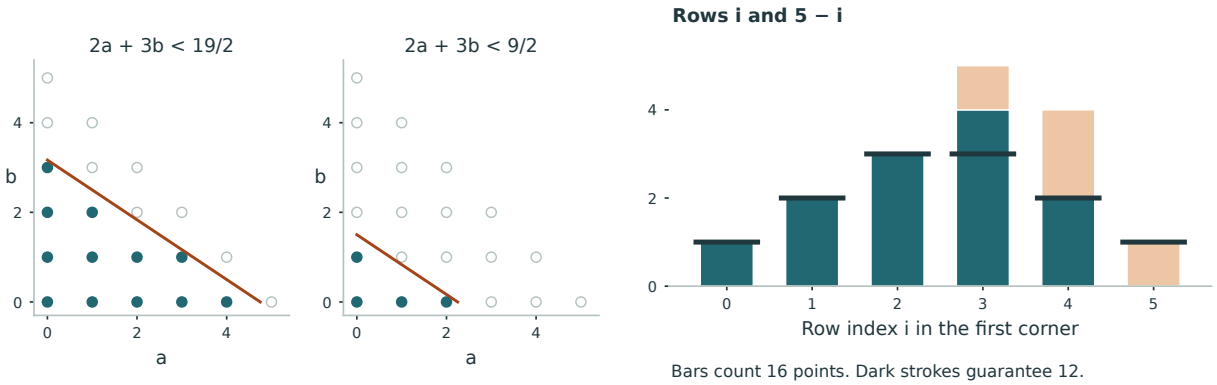


Figure 2: Filled dots satisfy the strict triangle inequality. For $S = 5$ the left triangle supplies $1, 2, 3, 4, 2, 0$ points in increasing row order. The right supplies $0, 0, 0, 1, 2, 1$ in reverse order. Each bar clears its dark stroke even if the two thresholds change while their sum stays 14.

their faces because (4) gives $\alpha_k \le hp_k$, so every contributing chain starts in exactly one such face. At the other end, a last lower endpoint below $t - 1$ gives the reflected triangle with threshold β_k. The two ways of missing a chain are disjoint. We have proved

$$C - \#\text{cuts} \;\ge\; \sum_{k=1}^{3} \big(\Delta(p_k, q_k; \alpha_k) + \Delta(p_k, q_k; \beta_k)\big). \tag{5}$$

The sum $\gamma_k = \alpha_k + \beta_k = 2hs_k - n$ depends only on the direction.

3.1 Reading an offset-free bound from two triangles

Write $T(m) = \lfloor (m+1)^2/4 \rfloor$ for $m \ge 1$, and $T(m) = 0$ for $m \le 0$. The sum of $\min(i, m+1-i)$ over $1 \le i \le m$ is $T(m)$. The same sum, with every entry capped at height L, is $T(m) - T(m - 2L)$.

Lemma 3. *Fix $0 < p < q$, thresholds α, β and an integer $S \ge 0$. Put $d = q - p$, $\gamma = \alpha + \beta$, and*

$$L = \min\big(\lfloor S/2 \rfloor + 1,\, \max(0, \lceil (\gamma - pS)/d \rceil)\big).$$

Then $\Delta(p, q; \alpha) + \Delta(p, q; \beta) \ge G(S)$, where $G(S) = T(S + 1) - T(S + 1 - 2L)$.

Proof. On the row $i + j = r$, the tested weights are $pr, pr + d, \ldots, pr + rd$. Its accepted count is $\lceil (\alpha - pr)/d \rceil$ clipped to $[0, r + 1]$. Pair this row with $i + j = S - r$ in the other triangle. Before rounding, the quotients add to $(\gamma - pS)/d$. If a row is full, it alone contributes at least $\min(r + 1, S - r + 1, L)$. If neither is full, rounding up and replacing negative counts by zero can only increase the combined lower bound. Summing the minima for $0 \le r \le S$ gives $G(S)$. For $p = q$, the same argument uses $L = (\lfloor S/2 \rfloor + 1)\mathbf{1}_{\{\gamma > pS\}}$: at least one paired row is complete whenever that indicator is one. □

Let $B_k = \max_{S \ge 0} G_k(S)$ for the kth pair (large S give zero). By (5) we can finish whenever $\sum B_k \ge D$. The maximum at a single level is $T(S + 1)$. Call that level full, put $\Gamma = 2\gamma/(p + q)$ and $\delta = d/(p + q)$. The test for fullness is

$$S - \mathbf{1}_{\{S \text{ odd}\}}\delta < \Gamma. \tag{6}$$

Indeed, $\lceil(\gamma - pS)/d\rceil \geq \lfloor S/2\rfloor + 1$ is equivalent to $\gamma > p\lceil S/2\rceil + q\lfloor S/2\rfloor$. When $S \geq \Gamma$ and $d > 0$, put $z = (S - \Gamma)/\delta$. An incomplete level loses at most

$$T(S+1) - G(S) \;\leq\; (z+2)^2/4. \tag{7}$$

To see the rounding explicitly, write $e = \mathbf{1}_{\{S \text{ odd}\}}$ and $f = \lfloor (z-e)/2\rfloor + 1$. The cap loses at most f units of height, and the part cut off weighs at most $f(f+e)$. This is at most $(z+2)^2/4$ in either parity.

3.2 The one chain that parity hides

Use the two slope differences $u = x_2 - x_1$, $w = x_3 - x_2$. Then

$$3(x_1, x_2, x_3) = (1 - 2u - w,\ 1 + u - w,\ 1 + u + 2w), \qquad (d_1, d_2, d_3) = (w, u + w, u). \tag{8}$$

Direct substitution of the parity formulas in Proposition 2 shows that for $n \geq 5$,

$$T_* = T(h-2) + 2T(h-3), \qquad D = T_* + \mathbf{1}_{\{n \text{ even}\}}. \tag{9}$$

For even $n = 2c + 4$, group the directions satisfying both

$$(3c+8)u - (3c+5)w < 1, \qquad (3c+5)u + (6c+13)w < 1 \tag{10}$$

in a region $K(c)$. Near the diagonal the three integer row bounds can stop one chain short. The same height t was used in all six corners. That restriction recovers the last chain.

Lemma 4. *Let $c \geq 1$, $u, w \geq 0$, $(u, w) \neq (0, 0)$ and $n = 2c + 4$. In $K(c)$ the right side of (5) is at least $T_* + 1$, for every t.*

Proof. By (8), the second inequality of (10) reads $\gamma_3 > (2c-1)x_1 + (c-1)u$ and the first reads $\gamma_2 > (2c-1)x_1 + c(u+w)$. Thus level $2c-1$ is full in both pairs. Since $s_1 > 2/3$, pair one is full at $2c$. Their combined contribution is $(c+1)^2 + 2c(c+1) = T_*$.

Put $\theta = t - n/2$, so the two thresholds in pair k are $\gamma_k/2 \pm \theta$. Define

$$a = \gamma_1/2 - cx_3 = (c+2)u + w, \qquad b = cx_1 - \gamma_2/2 = u - (c+1)w.$$

Here $a > 0$ and $a > b$. If $|\theta| < a$, both middle rows in pair one accept all $c+1$ points. Their combined bound had used only $c+1$. If $|\theta| \geq a$, then $|\theta| > b$. Exchange the two corners in pair two if necessary so its row of length $c+1$ accepts at least one point. Pair this row with the row of length c. If either is full, the two contribute at least $c+1$ (using that accepted point if the shorter row is full). Otherwise their untruncated thresholds sum to $(\gamma_2 - (2c-1)x_1)/d_2 > c$, so they again contribute at least $c+1$. Their previous guarantee was c. Every other row keeps its guarantee. □

3.3 How close to the diagonal can a competitor be?

The vertical-column bound already gives $u + 2w = 3x_3 - 1 < 1/2$. For large n we need the admissible directions to shrink as n grows. The loss in the continuous approximation is quadratic in the slope deviation.

Proposition 5. *For $n \geq 16$, any plane cutting at least $F(n)$ cells has*

$$u + 2w \leq 3r, \qquad r = \sqrt{\frac{8}{27(n+6)}}.$$

Every plane cuts at most $U(n) = \lfloor (9n^2 + 6n - 20)/4\rfloor$ for $n \geq 2$.

Proof. For $p \le q$, summing the j-rows of a positive-threshold triangle gives

$$\Delta(p,q;b) \ge b^2/(2pq) + b/(2p) \ge 2\big(b/(p+q)\big)^2 + b/(p+q).$$

Indeed, with $m = \lceil b/q \rceil$ the first row sum before rounding is $(mb - qm(m-1)/2)/p$, whose excess over the first bound is $q(m - b/q)(1 - m + b/q)/(2p) \ge 0$. For arbitrary b use its positive part on the right. By convexity, the two triangles for $s = s_k$ contribute at least $\psi(y) = A_+^2 + A_+$, where $y = 1/s$, $A = 2h - ny$, and $A_+ = \max(A, 0)$.

Let $a_0 = (h-1)(h-3)/4$ and put $\ell(y) = a_0 - \frac{9}{4}n(n-3)(2/3 - 1/y)$. If $A \ge 0$, expansion gives

$$\psi(y) - \ell(y) = \frac{n(2y-3)^2(ny - h + 2)}{4y} \ge 0. \tag{11}$$

For $A \le 0$ the identity $-4ny\ell(y) = (5h^2 - 2h - 15)(ny - 2h) + (h-3)^2(h+2)$ again gives $\psi \ge \ell$ when $h \ge 3$. The three linear terms cancel because $\sum_k s_k = 2$. Consequently the upper bound is $C - 3a_0 = (9n^2 + 6n - 20)/4$. For the column case $2n^2 \le U(n)$ when $n \ge 3$, and $m(2) = U(2) = 7$. The direct $n = 3$ argument gives $m(3) = U(3) = 19$.

A cut of size at least $F(n)$ leaves at most $n/2$ for the entire excess $\sum_k (\psi(1/s_k) - \ell(1/s_k))$. At $y = 1/s_3$ we have $3/2 \le y < 2$. If $ny \ge 2h$ its excess is at least $(h-3)^2(h+2)/(8n) > n/2$ for $h \ge 15$ (the polynomial difference is positive at 15 and increasing). Thus (11) applies. It gives $(2y-3)^2(ny - h + 2) \le 2y$. Use $4y(ny - h + 2) - 3(n+6) = (2y-3)(2ny + n + 6) \ge 0$ and $x_3 - 1/3 = (2y-3)/(3y)$ to obtain the stated radius. □

Proposition 6. *For $n \ge 700$ and $u, w > 0$ with $u + 2w \le 3r$, either $B_1 + B_2 + B_3 \ge D$, or n is even and $(u, w) \in K((n-4)/2)$.*

For a picture of the large-size argument, set $\Gamma_k = 2\gamma_k/s_k$ and start each triangle pair at the last level strictly below Γ_k. Those three levels cost nothing in (7). When their combined height falls short, increase one or two levels by one. The price is quadratic in the rounding error divided by the slope difference. The tilt also increases the sum of squares of the three levels, providing the budget to pay that price. Appendix A proves the uniform inequalities. Together with Lemma 4 and Proposition 5 this settles every $n \ge 700$.

4 A finite certificate over all real slopes

For a fixed n, the slope cone $Z(R) = \{u, w \ge 0 : u + 2w \le R\}$ is a compact triangle. Take $R = 1/2$ for $n < 16$. For $n \ge 16$ any rational $R \ge 3r$ suffices provided $R \le 1/2$. At each point it is enough to certify three row levels with combined bound at least D, or the two inequalities (10). For $1 \le l \le \lfloor S/2 \rfloor + 1$ the test that a row level contributes at least $T(S+1) - T(S+1-2l)$ is

$$\gamma_k - p_k S - (l-1)d_k > 0. \tag{12}$$

This follows from Lemma 3, including $d_k = 0$ on a coordinate axis. For fixed S, l, (8) makes it an affine test in u, w. The two coupling tests are affine as well.

A partition into regions. Starting with the three inequalities for the cone, split a region by a line until one of the two alternatives above can be certified there. To certify a strict affine inequality $H > 0$ on a leaf, exhibit a coefficient identity

$$mH = c + \sum_i \lambda_i H_i, \qquad m > 0, \quad c, \lambda_i \ge 0,$$

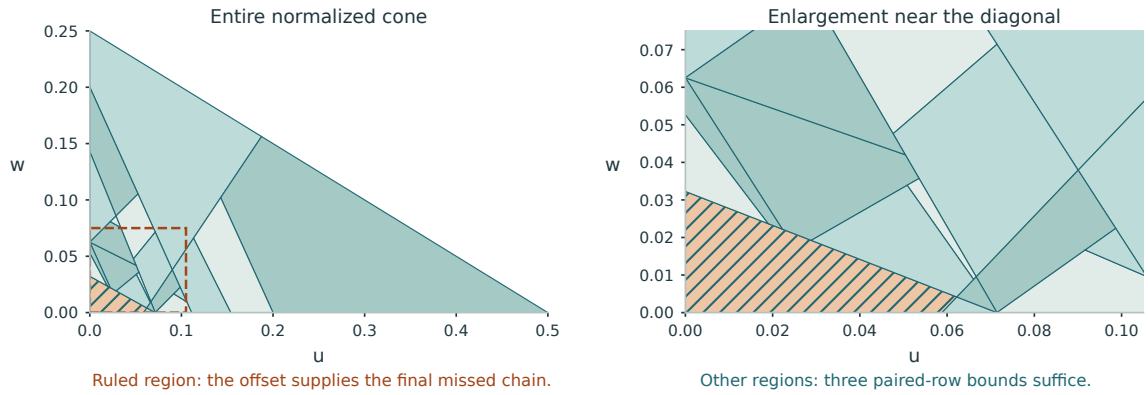


Figure 3: The actual $n = 10$ proof tree cut into its 21 leaves. A point is a normalized direction, with $u = x_2 - x_1$ and $w = x_3 - x_2$. In 20 leaves three row levels certify $D = 41$ missed chains. In the hatched leaf the common height gives the bound through Lemma 4. The dashed rectangle is enlarged at right. Boundaries inherit the strict or weak side specified by the tree.

where the recorded H_i are nonnegative in the region, and either $c > 0$ or a positive coefficient multiplies a strict H_i. Clearing denominators makes every identity integral. At each split the daughters take $H > 0$ and $H \le 0$. An optional split at $u = w$ can take two strict sides, since the perturbation above guarantees $u \neq w$. Induction from the leaves proves the claim everywhere that an improving plane could occur. No counting of sampled directions is involved.

The 28 trees for $5 \le n \le 32$ have 1,299 leaves in total, 235 at sizes 5 through 15 and 1,064 at sizes 16 through 32. Figure 3 gives one of them in its entirety.

Testing vertices instead. Choose any finite family of the tests (12), together with the two coupling tests when appropriate. Draw their zero lines and the edges of $Z(R)$. Suppose every nonzero intersection in the cone of two nonparallel lines admits one of the sufficient alternatives. Then so does every nonzero point of the cone. To prove this, fix a point $p \neq 0$ and impose weakly all inequalities $H \le 0$ for tests false at p. The resulting bounded polygon (possibly a segment or a point) contains p and has a nonzero vertex v. Every test true at v is true at p. The choices of levels or coupling tests that certify v therefore certify p. Edges, triple intersections and ties require no general-position assumption. A narrow family of levels may fail to prove the target. A successful family proves it for every direction.

Two independent C implementations use this argument on $16 \le n \le 700$. Both compute rational intersections by determinants and compare integers after clearing the common denominator. They choose different level windows and cone radii. The integer inequality $3(n+6)R_p^2 \ge 8R_q^2$ certifies $R = R_p/R_q \ge 3r$; integer comparisons also verify the window bounds. Their counts below include multiplicities when several line pairs meet at the same point.

Implementation	sizes	in-cone intersections tested	failed windows
First	685	335,094,616	0
Second	685	402,548,904	0

Both give minimum certified margin zero at every size. For example, at $n = 16$ and $(u, w) = (0, 1/49)$ the three full levels $12, 12, 10$ supply $49 + 49 + 36 = 134 = D$. The kernel-checked trees

already cover this size. The only external obligations needed by Theorem 1 are $33 \le n \le 699$. The two vertex runs also cover that entire remaining interval. This completes the proof of the upper bound.

5 What the formal proof checks

The Lean 4 library `CheeseCutM` encodes a cut by the same open-interval test used here. The verification record for this draft's underlying library revision `6ea439be` covers 164 public lemmas and theorems, with no `sorry` and no `native_decide`. The recorded axioms are propositional extensionality, classical choice and quotient soundness. The exact values through size 32 and the theorem for all $n \ge 700$ use the kernel. The finite trees are evaluated by kernel reduction.

The all-sizes declaration in `SmallNAndDim.lean` takes an argument asserting passage of the arrangement check for every $33 \le n \le 699$. The real-geometric implication from those finite checks to the unrestricted theorem is kernel-proved. The two programs establish the finite condition externally. Replaying the arrangements with a second implementation and checking the sufficiency theorem in Lean answer different questions.

6 Other dimensions and open questions

The interior convention leads to a sharp result in the plane with no deficit: there are $2n-1$ diagonal chains, and $x + y = n - 1/2$ meets squares with $i + j = n - 2$ or $n - 1$, altogether $2n - 1$. In three dimensions the six face triangles measure the obstruction to that same argument.

For standard arithmetic planes in the sense of Andres [1], fix integral $v \ge 0$, $W = \sum v_i > 0$ and $\mu \in \mathbb{Z}$. The points satisfying $\mu \le v \cdot z < \mu + W$ in the index box are exactly the cells cut at $t = \mu + W - 1/2$, by integrality of $v \cdot z$. Hence the same extremum holds for these digital planes.

Bárány and Frankl established an asymptotic formula for hyperplane cuts in general dimension [3]. A finite-grid correction can even reverse the preferred tilt. In $[0, 3]^4$ the best translates for normals $(1, 1, 1, 1)$ and $(2, 2, 2, 3)$ cut 61 and 62 cells, respectively. Yet the normal $(1, 2, 2, 2)$ at height $21/2$ cuts 63. One direct check multiplies $\prod_i (1 + z^{v_i} + z^{2v_i})$. Its coefficients count lower endpoints by height, and the largest sum of W consecutive coefficients gives the maximum over translates.

The 63 are optimal over all normals. There are $3^4 - 2^4 = 65$ diagonal chains. Normalize and sort the four weights. Pair the one-cell chains $(0, 2, 0, 0)$ and $(0, 2, 2, 2)$, then take the second pair obtained by interchanging the first two coordinates. Within either pair the lower endpoints differ by $2(x_3 + x_4) \ge 1$, so at least two chains are missed. This is the same obstruction that gave $m(4) = 35$ in three dimensions. The separate `CheeseCut4D` library also proves this small exact value.

Which normals maximize the count at larger sizes in four dimensions? In three dimensions, can the finite middle interval be replaced by a uniform argument, and which other planes realize the optimum?

7 Data, reproduction and AI assistance

The accompanying repository contains the figures, the exact witness check, the two arrangement implementations and their per-size records. From the repository root, run `python3`

`papers/cheese-cut/draft3/recompute.py`. The figure program computes the polygonal sections, the accepted lattice points, and the normalized regions directly from the finite certificate. The reference ledger records the primary texts and the sections used. AI assisted with mathematical exploration, proof formalization and the preparation of this private author-review draft.

A Paying for one or two rounded levels

This appendix proves Proposition 6. Its two parameters have different jobs. The size $X = n(u+w)$ measures tilt in lattice units. The quantity E below measures the curvature lost when the three slopes are moved away from their common diagonal value. Small E forces the three rounding errors to share a single budget.

A.1 An accounting identity

Put $\eta_k = s_k - 2/3$ and $\kappa_k = s_k/d_k$. With $\Gamma_k = h - 3 + g_k$ the definitions give

$$g_k = 3n\eta_k/s_k, \quad j_k = \lceil g_k \rceil, \quad J_k = h - 3 + j_k, \quad \varepsilon_k = j_k - g_k, \quad z_k = \kappa_k \varepsilon_k.$$

For the rest of the appendix $n \geq 700$, $u, w > 0$, $u + 2w \leq 3r$. In particular $r < 0.0204862$, $0.64618 \leq s_k \leq 0.70764$ and $d_2 \leq 3r$. All J_k are positive because $\Gamma_k = 4h - 2n/s_k > 0$ on these bounds. Write

$$E = -\sum_k g_k, \qquad N = \sum_k j_k, \qquad Q = \sum_k g_k^2 - 2(h-2)E.$$

Since $\sum \eta_k = 0$, the identity $\eta_k/s_k = \frac{3}{2}\eta_k - \frac{3}{2}\eta_k^2/s_k$ gives

$$E = \frac{9n}{2}\sum_k \frac{\eta_k^2}{s_k} > 0, \qquad Q = 9n\sum_k \frac{\eta_k^2}{s_k}\left(\frac{n(1-s_k)}{s_k} + 3\right) \geq 0, \qquad \sum_k \varepsilon_k = N + E. \tag{13}$$

The largest level strictly below Γ_k is $J_k - 1$, and (6) makes it full. At level J_k the loss $c_k = T(J_k+1) - G_k(J_k)$ is an integer with $4c_k \leq (z_k+2)^2$ by (7). Our three candidates for $\sum B_k$ are $V_0 = \sum T(J_k)$, V_a obtained by raising component a and subtracting c_a, and V_{ab} obtained by raising a, b and subtracting both losses. Thus every candidate is an actual integer lower bound.

For $M_k = h - 3 + \mu_k$ put $\pi(M) = \#\{k : M_k \text{ even}\} - e_*$, where $e_* = 1$ for odd n and 2 for even n. Expanding $4T(m) = (m+1)^2 - \mathbf{1}_{\{m \text{ even}\}}$ gives, at these levels,

$$4\left(\sum T(M_k) - T_*\right) = 2(h-2)\left(\sum \mu_k - 1\right) + \sum \mu_k^2 - 1 - \pi(M). \tag{14}$$

Applying (14) at $J + e_a$ and $J + e_a + e_b$ (where e_a is the ath unit vector) yields respectively

$$4(V_a - T_*) + \pi(J + e_a) \geq Q + 2g_a - 4 + \varepsilon_a(2\Gamma_a + 4 - 4\kappa_a - \kappa_a^2\varepsilon_a), \tag{15}$$

$$4(V_{ab} - T_*) + \pi(J + e_a + e_b) \geq 2(h-2) + Q + 2g_a + 2g_b - 7 - z_a^2 - 4z_a - z_b^2 - 4z_b. \tag{16}$$

For example, substitute $j_i = g_i + \varepsilon_i$ before discarding the nonnegative terms $2(h - 2 + g_i)\varepsilon_i + \varepsilon_i^2$. In (15) keep the linear ε_a term and the extra $2\varepsilon_a$ from raising that level. Since $\pi \leq 2$ at odd n and $\pi \leq 1$ at even n, a right side greater than one proves $V \geq D$ by integrality and (9).

When the full levels already suffice. Use (14) with $\mu = j$. For $N \geq 2$ its right side is positive. For $N = 1$ and odd n, $1 + \pi(J)$ is the number of odd j_k, bounded by $\sum j_k^2$. For $N = 1$ and even n the result is positive unless $j = (1, 0, 0)$: here $4(V_0 - T_*) = \sum j_k^2 + 1 - \#\{k : j_k \text{ even}\}$ and $g_1 > 0 > g_3$. In the exceptional case write $n = 2c + 4$. The inequality $g_3 > -1$ is precisely the second condition of (10) after substitution. Failure of its first condition gives $n(u-w) \geq 2x_1$. Also $g_1 \leq 1$ gives $s_1 \geq n(2u + w) \geq 2n(u - w) \geq 4x_1$. Then $x_1 \leq 1/5$ and $x_3 \geq (1 - x_1)/2 \geq 2/5$, contradicting $x_3 \leq 1/3 + r$ with $r < 1/15$. This settles $N \geq 1$.

A.2 When one full level has to be bought

Assume $N \leq 0$. Write $d = u + w$, $X = nd$, $\phi = \eta_2/d = (w - u)/(3d)$, and $H = 1 + 3\phi^2$. We have $|\phi| \leq 1/3$, $1 \leq H \leq 4/3$ and $\sum \eta_k^2 = Hd^2/2$. Substituting the preceding bounds for s_k into (13) gives the following one-sided estimates. All terminating decimals below denote exact rationals.

$$\begin{aligned} Q &\geq 2.627HX^2, & 3.1795HX^2/n \leq E &\leq 3.4821HX^2/n, \\ |g_2| &\leq 1.5476X, & \kappa_2 &\leq 0.70764n/X. \end{aligned} \tag{17}$$

For instance the coefficient of HX^2 in the lower bound for Q is at least $9(1-0.70764)/(2\cdot 0.70764^2)$. One estimate must retain its leading constant 2:

$$\kappa_2^2 E \leq n(2 + 4.295d). \tag{18}$$

Indeed the left side divided by n is at most $9Hs_2^2/(4s_3)$. With $s_2 = 2/3 + \phi d$ and $s_3 = 2/3 - (1 + \phi)d/2$, expansion gives

$$\begin{aligned} d(8 + d)/3 + 2s_3 - 9Hs_2^2/4 = 1/3 - 3\phi^2 + d(1/3 - \phi)(5 + 3\phi + 9\phi^2) \\ + d^2(1/3 - 9H\phi^2/4) \geq 0. \end{aligned}$$

The coefficient $(8 + d)/(3s_3)$ is below 4.295 on our bounds.

If $E \geq 1$. Use (15) with $a = 2$. The last term is concave in ε_2, so it is bounded below by its smaller endpoint value for $0 \leq \varepsilon_2 \leq 1$. Write $2\Gamma_2 + 4 = 2n - 4 + 2g_2$. From (18), $(\kappa_2^2 - 2n)_+ \leq 4.295X$. Thus the right side is at least

$$2.627HX^2 - 10.486X - 8 - 2.8306n/X.$$

This expression increases for $X \geq 2$, and (17) requires $X \geq X_0 = \sqrt{0.28718n/H}$. Here $0.4640\sqrt{n} \leq X_0 \leq 0.5359\sqrt{n}$. Substitution gives a lower bound $0.7544n - 11.72\sqrt{n} - 8$, increasing for $n \geq 700$ and already greater than 209 there.

If $E < 1$. Now $N = 0$ because $\sum \varepsilon_k = N + E \geq 0$ and N is an integer. Consequently $\varepsilon_k \leq E$. Also (17) gives $X < 0.56082\sqrt{n}$ and $d < 0.0212$. Substitution in $s_1 = 2/3 + (1 - \phi)d/2$, $s_2 = 2/3 + \phi d$, $s_3 = 2/3 - (1 + \phi)d/2$ gives

$$s_1 \leq 0.6809, \quad s_3 \geq 0.6525, \quad 0.6595 \leq s_2 \leq 0.6738.$$

Using these endpoints in the same inequalities sharpens them to

$$Q \geq 3.097HX^2, \quad E \leq 3.4483HX^2/n, \quad \kappa_2^2 E \leq 2n + 4.2X, \quad \kappa_2 \leq 0.6738n/X.$$

For $X \ge 6$, take the concavity endpoints $0, E$ in (15). The possible losses at E include $E(4.2X + 4) \le 0.3268HX^2$ and $4E\kappa_2 \le 9.294HX$. For $\phi \ge 0$ the remaining g_2 terms are nonnegative. The right side of (15) is therefore at least $H(2.7702X^2 - 9.294X) - 4 > 39$. For $\phi = -\tau < 0$, $g_2 \ge -4.549\tau X$ and $2E|g_2| \le 0.2218HX^2$. After collecting terms we get

$$H\big(2.5484X^2 - 9.294X\big) - 9.098\tau X - 4 \;\ge\; 2.5484X^2 - 12.327X - 4 \;\ge\; 13.7.$$

Both endpoint minima thus exceed one.

A.3 When two cheap levels finish the proof

It remains to handle $E < 1$ and $X < 6$. A single component may have a tiny d_k and hence an expensive rounding loss. Raise component two, whose $d_2 = u + w$ is largest, and whichever of $b = 1, 3$ minimizes z_b/s_b. Since $d_1 + d_3 = d_2$, their weighted average gives

$$z_2/s_2 + z_b/s_b \le (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)/d_2 = E/d_2 \le 3X/s_3.$$

The last bound uses (13) and $H \le 4/3$. Hence $z_2 + z_b \le 18(0.6809/0.6525) < 18.79$, and the combined penalty $z_2^2 + 4z_2 + z_b^2 + 4z_b$ is below 428.3. Also $g_2 \ge -9.098$, $g_3 \ge -18.391$, $g_1 > 0$, by their defining formulas. The right side of (16) exceeds $2(h-2) - 54.98 - 7 - 428.3 \ge 903.72$. Every case has supplied the required integer candidate or the coupling region. This proves Proposition 6.

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